THE SPECTRUM OF TWISTED DIRAC OPERATORS ON COMPACT FLAT MANIFOLDS

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Abstract. Let M be an orientable compact flat Riemannian manifold endowed with a spin structure. In this paper we determine the spectrum of Dirac operators acting on smooth sections of twisted spinor bundles of M, and we derive a formula for the corresponding eta series. In the case of manifolds with holonomy group $\mathbb{Z}_2^k$, we give a very simple expression for the multiplicities of eigenvalues that allows to compute explicitly the η-series in terms of values of Riemann-Hurwitz zeta functions, and the η-invariant. We give the dimension of the space of harmonic spinors and characterize all $\mathbb{Z}_2^k$-manifolds having asymmetric Dirac spectrum.

Furthermore, we exhibit many examples of Dirac isospectral pairs of $\mathbb{Z}_2^k$-manifolds which do not satisfy other types of isospectrality. In one of the main examples, we construct a large family of Dirac isospectral compact flat n-manifolds, pairwise non-homeomorphic to each other.

Introduction

The relation between the geometry of a compact Riemannian manifold M and the spectral properties of the Laplace operators, Δ and Δ_p, acting respectively on smooth functions or on smooth p-forms, has been studied extensively. An elliptic differential operator whose spectrum is less understood is the Dirac operator D. It can be defined for Riemannian manifolds having an additional structure, a spin structure. The goal of the present paper is to investigate properties of the spectrum of Dirac operators on a compact flat manifold M and with a flat twist bundle. We compare it to other spectral or geometric properties of M.

The spectrum of the Dirac operator is explicitly known for a small class of Riemannian manifolds (see for instance [**Ba**] or the list in [**AB**], Table 1). In the context of flat manifolds, Friedrich ([**Fr**]) determined the spectrum of D for flat tori $T_\Lambda = \Lambda\backslash\mathbb{R}^n$, Λ a lattice in $\mathbb{R}^n$, showing the dependence on the spin structure. In [**Pf**] Pfäffle studied in detail the Dirac spectrum of 3-dimensional orientable compact flat manifolds, determining the eta invariants.

The goal of this paper is to study the spectrum of D and the eta series for an arbitrary compact flat manifold. Such a manifold is of the form

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$M_\Gamma := \Gamma\backslash\mathbb{R}^n$, Γ a Bieberbach group. If Λ denotes the translation lattice of Γ, then $F = \Lambda\backslash\Gamma$ is a finite group, the holonomy group of M_Γ. We note that by the Cartan-Ambrose-Singer theorem any Riemannian manifold with finite holonomy group is necessarily flat. As in [**Ch**], we shall use the terminology F-manifold for a Riemannian manifold with finite holonomy group F. Already the case when $F \simeq \mathbb{Z}_2^k$ provides a very large class of manifolds with a rich combinatorial structure (see Section 3).

The spin structures on M_Γ are in one to one correspondence with homomorphisms $\varepsilon : \Gamma \to \mathrm{Spin}(n)$ satisfying $\mu \circ \varepsilon = r$ (see Section 1). In [**MP**] we have shown one can not hear the existence of spin structures on flat manifolds. For this purpose, we obtained necessary and sufficient conditions for the existence of spin structures on $\mathbb{Z}_2^k$-manifolds. We note that Vasquez has already shown in [**Va**] that not every flat manifold admits a spin structure by giving some examples of M with holonomy group $\mathbb{Z}_2^k$ having nonzero second Stiefel-Whitney class, $w_2(M) \neq 0$ (see also [**IK**] and [**LS**]).

Every unitary complex representation $\rho : \Gamma \to \mathrm{U}(V)$, defines a flat vector bundle $E_\rho := \Gamma\backslash(\mathbb{R}^n \times V)$ over M_Γ. Now, if (L, S) denotes the spin representation of $\mathrm{Spin}(n)$ we can consider the associated twisted spinor bundle $S_\rho(M_\Gamma, \varepsilon) := \Gamma\backslash(\mathbb{R}^n \times (\mathrm{S} \otimes V))$ over M_Γ, where $\gamma \cdot (x, w \otimes v) = (\gamma x, L(\varepsilon(\gamma))(w) \otimes \rho(\gamma)(v))$, for $w \in \mathrm{S}, v \in V$. We denote by D_ρ, the Dirac operator acting on smooth sections of $S_\rho(M_\Gamma, \varepsilon)$. For simplicity, we shall assume throughout this paper that $\rho_{|\Lambda} = Id$, that is, ρ induces a complex unitary representation of the holonomy group $F \simeq \Lambda\backslash\Gamma$. Similar results could be derived in more generality, for instance, assuming that $\rho_{|\Lambda} = \chi$, χ an arbitrary unitary character of Λ. However, this would make the statements of the results more complicated, without gaining too much in exchange.

We now describe the main results. In Theorem 2.5, for an arbitrary compact flat manifold M_Γ, we obtain formulas for the multiplicities $d^{\pm}_{\rho,\mu}(\Gamma, \varepsilon)$ of the eigenvalues $\pm 2\pi\mu$, $\mu > 0$, of the Dirac operator D_ρ associated to a spin structure ε, in terms of the characters χ_ρ of ρ and χ_{L_n}, $\chi_{L_{n-1}^{\pm}}$ of the spin and half spin representations. The multiplicity formula reads as follows. If n is odd then

$$d^{\pm}_{\rho,\mu}(\Gamma, \varepsilon) = \tfrac{1}{|F|} \Big(\sum_{\substack{\gamma \in \Lambda\backslash\Gamma \\ B \notin F_1}} \chi_\rho(\gamma) \sum_{u \in (\Lambda^*_{\varepsilon,\mu})^B} e^{-2\pi i u\cdot b} \, \chi_{L^{\pm}_{n-1}}(x_\gamma) + \tag{0.1}$$
$$\sum_{\substack{\gamma \in \Lambda\backslash\Gamma \\ B \in F_1}} \chi_\rho(\gamma) \sum_{u \in (\Lambda^*_{\varepsilon,\mu})^B} e^{-2\pi i u\cdot b} \, \chi_{L_{n-1}^{\pm\sigma(u,x_\gamma)}}(x_\gamma) \Big).$$

Here F_1 is the subset of F corresponding to the elements $BL_b \in \Gamma$ with $n_B := \dim\ker(B - Id) = 1$ and $\Lambda^*_{\varepsilon,\mu} = \{u \in \Lambda^* : \|u\| = \mu, \varepsilon(\lambda) = e^{2\pi i \lambda\cdot u}$ for every $\lambda \in \Lambda\}$. Furthermore, for $\gamma \in \Gamma$, x_γ is an element in the

maximal torus of $\mathrm{Spin}(n-1)$ conjugate in $\mathrm{Spin}(n)$ to $\varepsilon(\gamma)$, and $\sigma(u, x_\gamma)$ is a sign depending on u and on the conjugacy class of x_γ in $\mathrm{Spin}(n-1)$.

If n is even, then the formula reduces to the first summand in (0.1) with $\chi_{L^{\pm}_{n-1}}$ replaced by $\chi_{L_{n-1}}$.

We also compute the dimension of the space of harmonic spinors, (see (2.10)), showing that these can only exist for a special class of spin structures, namely, those restricting trivially to the lattice of translations Λ.

As a consequence of the theorem, we give an expression, (2.18), for the η-series $\eta_{(\Gamma,\rho,\varepsilon)}(s)$ corresponding to D_ρ acting on sections of $S_\rho(M_\Gamma, \varepsilon)$.

In Sections 3 and 4, we restrict ourselves to the case of $\mathbb{Z}_2^k$-manifolds. In this case, by computing $\chi_{L_{n-1}}$ and $\chi_{L^{\pm}_{n-1}}$ we give very explicit expressions for the multiplicities. Indeed, if the spectrum is symmetric, we have that $d^{\pm}_{\rho,\mu}(\Gamma, \varepsilon) = 2^{m-k-1}\, d_\rho\, |\Lambda^*_{\varepsilon,\mu}|$ for each $\mu > 0$.

The spin $\mathbb{Z}_2^k$-manifolds (M_Γ, ε) having asymmetric Dirac spectrum are of a very special type. This happens if and only if $n = 4r + 3$ and there exists $\gamma = BL_b \in \Gamma$, with $n_B = 1$ and $\chi_{_\rho}(\gamma) \neq 0$, such that $B_{|\Lambda} = -\delta_\varepsilon Id$. In this case, the asymmetric spectrum is the set of eigenvalues

$$\{\pm 2\pi\mu_j : \mu_j = (j + \tfrac{1}{2})\|f\|^{-1},\, j \in \mathbb{N}_0\}$$

where f satisfies $\Lambda^B = \mathbb{Z}f$. We furthermore have:

$$d^{\pm}_{\rho,\mu}(\Gamma, \varepsilon) = \begin{cases} 2^{m-k-1}\left(d_\rho\, |\Lambda^*_{\varepsilon,\mu}| \pm 2\sigma_\gamma (-1)^{r+j} \chi_{_\rho}(\gamma)\right) & \mu = \mu_j, \\ 2^{m-k-1}\, d_\rho\, |\Lambda^*_{\varepsilon,\mu}| & \mu \neq \mu_j \end{cases}$$

where $\sigma_\gamma \in \{\pm 1\}$. We also give an explicit expression for the eta series:

$$\eta_{(\Gamma,\rho,\varepsilon)}(s) = (-1)^r\, \sigma_\gamma\, \chi_{_\rho}(\gamma)\, 2^{m-k+1}\, \frac{\|f\|^s}{(4\pi)^s}\big(\zeta(s, \tfrac{1}{4}) - \zeta(s, \tfrac{3}{4})\big)$$

where $\zeta(s, \alpha) = \sum_{j=0}^{\infty} \frac{1}{(j+\alpha)^s}$ denotes the generalized Riemann-Hurwitz zeta function for $\alpha \in (0, 1]$. From this we obtain that the η-invariant of M_Γ equals $\eta_\rho = \pm\chi_{_\rho}(\gamma)\, 2^{[n/2]-k}$, the sign depending on ε. This generalizes a result in [**Pf**] in the case when $n = 3$. We summarize these results in Theorem 3.2 and Proposition 3.4.

In Section 4 we compare Dirac isospectrality with other types of isospectrality –see Table 1 below–; namely, isospectrality with respect to the spinor Laplacian $\Delta_{s,\rho} := -D^2_\rho$ and to the Hodge Laplacian on p-forms, Δ_p for $0 \le p \le n$. We also look at the length spectrum of M or $[L]$-*spectrum*, and the weak length spectrum of M or L-*spectrum*, that is, the set of lengths of closed geodesics counted with and without multiplicities, respectively.

The information in Examples 4.3, 4.4, 4.5 is collected in Theorem 4.1. We summarize the results in the following table that shows the independence of Dirac isospectrality from other notions of isospectrality considered.

Table 1. Isospectrality

D_ρ	$\Delta_{s,\rho}$	Δ_p $(0 \leq p \leq n)$	$[L]$	L	Ex.	dim
Yes	*Yes*	*No* (generically)	*No*	*No*	4.3 (i)	$n \geq 3$
Yes	*Yes*	*Yes* (if p odd)	*No*	*No*	4.3 (iii)	$n = 4t$
No	*Yes*	*No*	*No*	*No*	4.4 (i)	$n \geq 7$
Yes/No	*Yes/No*	*Yes* $(0 \leq p \leq n)$	*Yes*	*Yes*	4.5 (i)	$n \geq 4$
Yes/No	*Yes/No*	*Yes* $(0 \leq p \leq n)$	*No*	*Yes*	4.5 (ii)	$n \geq 4$

Finally, in Example 4.6, starting from Hantzsche-Wendt manifolds (see [**MR**]), we construct a large family (of cardinality depending exponentially on n or n^2) of $\mathbb{Z}_2^{n-1}$-manifolds of dimension $2n$, n odd, pairwise non-homeomorphic, and Dirac isospectral to each other.

In Section 5, by specializing our formula for $\eta(s)$, we give an expression for the eta series and eta invariant for a p-dimensional $\mathbb{Z}_p$-manifold, for each $p = 4r+3$ prime. We obtain an expression for the η-invariant involving Legendre symbols and trigonometric sums and give a list of the values for $p \leq 503$. If $p = 3$, our values are in coincidence with those in [**Pf**]. Our formulas for the η-series are reminiscent of those obtained in [**HZ**] to compute the G-index of elliptic operators for certain low dimensional manifolds. In the case when $F \simeq \mathbb{Z}_n$ in dimension $n = 4r+3$ (n not necessarily prime), an alternative expression for the η-invariant in terms of the number of solutions of certain congruences mod(n), has been given in [**SS**], with explicit calculations in the cases $n = 3, 7$.

1. Preliminaries

Bieberbach manifolds. We first review some standard facts on compact flat manifolds (see [**Ch**] or [**Wo**]). A *Bieberbach group* is a discrete, cocompact, torsion-free subgroup Γ of the isometry group $I(\mathbb{R}^n)$ of $\mathbb{R}^n$. Such Γ acts properly discontinuously on $\mathbb{R}^n$, thus $M_\Gamma = \Gamma\backslash\mathbb{R}^n$ is a compact flat Riemannian manifold with fundamental group Γ. Any such manifold arises in this way and will be referred to as a *Bieberbach manifold*. Any element $\gamma \in I(\mathbb{R}^n) = \mathrm{O}(n) \rtimes \mathbb{R}^n$ decomposes uniquely as $\gamma = BL_b$, with $B \in \mathrm{O}(n)$ and $b \in \mathbb{R}^n$. The translations in Γ form a normal maximal abelian subgroup of finite index L_Λ where Λ is a lattice in $\mathbb{R}^n$ which is B-stable for each $BL_b \in \Gamma$. The restriction to Γ of the canonical projection $r : \mathrm{I}(\mathbb{R}^n) \to \mathrm{O}(n)$ given by $BL_b \mapsto B$ is a homomorphism with kernel L_Λ and $F := r(\Gamma)$ is a finite subgroup of $\mathrm{O}(n)$. Thus, given a Bieberbach group Γ, there is an exact sequence $0 \to \Lambda \to \Gamma \xrightarrow{r} F \to 1$. The group $F \simeq \Lambda\backslash\Gamma$ is called the

holonomy group of Γ and gives the linear holonomy group of the Riemannian manifold M_Γ. Since we will be working with spin manifolds, we shall assume throughout this paper that M_Γ is orientable, i.e. $F \subset \mathrm{SO}(n)$. The action by conjugation of $\Lambda\backslash\Gamma$ on Λ defines an integral representation of F, called the *holonomy representation*. This representation can be rather complicated. For instance, already in the case when $F \simeq \mathbb{Z}_2^2$, there are Bieberbach groups with indecomposable holonomy representations for arbitrarily large n. By an F-manifold we understand a Riemannian manifold with holonomy group F. In Sections 3 and 4 of this paper we will study in detail the Dirac spectrum of $\mathbb{Z}_2^k$-manifolds.

Let $\Lambda^* = \{\lambda' \in \mathbb{R}^n : \lambda \cdot \lambda' \in \mathbb{Z} \text{ for any } \lambda \in \Lambda\}$ denote the dual lattice of Λ and, for any $\mu \geq 0$, let $\Lambda^*_\mu = \{\lambda \in \Lambda^* : \|\lambda\| = \mu\}$. We note that this Λ^*_μ equals $\Lambda^*_{\mu^2}$ in the notation used in [**MR4**], [**MR5**], in the study of the spectrum of Laplace operators.

If $B \in \mathrm{O}(n)$ set

$$(\Lambda^*)^B = \{\lambda \in \Lambda^* : B\lambda = \lambda\}, \qquad (\Lambda^*_\mu)^B = \Lambda^*_\mu \cap (\Lambda^*)^B,$$
$$n_B := \dim\ker(B - Id) = \dim(\mathbb{R}^n)^B. \tag{1.1}$$

If Γ is a Bieberbach group then the torsion free condition implies that $n_B > 0$ for any $\gamma = BL_b \in \Gamma$. Since B preserves Λ and Λ^* we also have that $(\Lambda^*)^B \neq 0$. For such Γ, we set

$$F_1 = F_1(\Gamma) := \{B \in F = r(\Gamma) : n_B = 1\}. \tag{1.2}$$

where n_B is as in (1.1).

Spin group. Let $Cl(n)$ denote the Clifford algebra of $\mathbb{R}^n$ with respect to the standard inner product $\langle\, , \rangle$ on $\mathbb{R}^n$ and let $\mathbb{C}l(n) = Cl(n) \otimes \mathbb{C}$ be its complexification. If $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$ then a basis for $Cl(n)$ is given by the set $\{e_{i_1} \ldots e_{i_k} : 1 \leq i_1 < \cdots < i_k \leq n\}$. One has that $vw + wv + 2\langle v, w\rangle = 0$ holds for all $v, w \in \mathbb{R}^n$, thus $e_ie_j = -e_je_i$ and ${e_i}^2 = -1$ for $i, j = 1, \ldots, n$. Inside the group of units of $Cl(n)$ we have the spin group given by

$$\mathrm{Spin}(n) = \{g = v_1 \ldots v_k : \|v_j\| = 1,\ j = 1, \ldots, k \text{ even}\}.$$

which is a compact, simply connected Lie group if $n \geq 3$. There is a Lie group epimorphism with kernel $\{\pm 1\}$ given by

$$\mu : \mathrm{Spin}(n) \to \mathrm{SO}(n), \qquad v \mapsto (x \to vxv^{-1}). \tag{1.3}$$

If B_j is a matrix for $1 \leq j \leq m$, we will abuse notation by denoting by $\mathrm{diag}(B_1, \ldots, B_m)$ the matrix having B_j in the diagonal position j. Let

$B(t) = \left[\begin{smallmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{smallmatrix}\right]$ with $t \in \mathbb{R}$ and for $t_1, \dots, t_m \in \mathbb{R}$ let

$$
\begin{aligned}
x_0(t_1,\dots,t_m) &:= \begin{cases} diag(B(t_1),\dots,B(t_m)), & \text{if } n = 2m \\ diag(B(t_1),\dots,B(t_m),1), & \text{if } n = 2m+1 \end{cases} \\
x(t_1,\dots,t_m) &:= \prod_{j=1}^{m} (\cos t_j + \sin t_j e_{2j-1}e_{2j}) \in \mathrm{Spin}(n).
\end{aligned} \tag{1.4}
$$

Maximal tori in $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$ are respectively given by

$$
T_0 = \Big\{ x_0(t_1,\dots,t_m) \;:\; t_j \in \mathbb{R} \Big\}, \quad T = \{x(t_1,\dots,t_m) : t_j \in \mathbb{R}\}. \tag{1.5}
$$

The restriction $\mu : T \to T_0$ is a 2-fold cover (see [**LM**]) and

$$
\mu(x(t_1,\dots,t_m)) = x_0(2t_1,\dots,2t_m).
$$

The Lie algebras of $\mathrm{Spin}(n)$ and that of T are $\mathfrak{g} = \mathrm{span}\{e_ie_j : 1 \le i < j \le n\}$ and $\mathfrak{t} = \mathrm{span}\{e_{2j-1}e_{2j} : 1 \le j \le m\}$, respectively.

In the Appendix we have collected some specific facts on spin groups and spin representations that are used in the body of the paper.

Spin structures on flat manifolds. If (M, g) is an orientable Riemannian manifold of dimension n, let $\mathrm{B}(M) = \bigcup_{x\in M} \mathrm{B}_x(M)$ be the bundle of oriented frames on M and $\pi : \mathrm{B}(M) \to M$ the canonical projection, that is, for $x \in M$, $\mathrm{B}_x(M)$ is the set of ordered oriented orthonormal bases $(v_1,\dots,v_n)$ of $T_x(M)$ and $\pi((v_1,\dots,v_n)) = x$. $\mathrm{B}(M)$ is a principal $\mathrm{SO}(n)$-bundle over M. A *spin structure* on M is an equivariant 2-fold cover $p : \tilde{\mathrm{B}}(M) \to \mathrm{B}(M)$ where $\tilde\pi : \tilde{\mathrm{B}}(M) \to M$ is a principal $\mathrm{Spin}(n)$-bundle and $\pi \circ p = \tilde\pi$. A manifold in which a spin structure has been chosen is called a *spin manifold*.

We will be interested on spin structures on quotients $M_\Gamma = \Gamma\backslash\mathbb{R}^n$, Γ a Bieberbach group. If $M = \mathbb{R}^n$, we have $\mathrm{B}(\mathbb{R}^n) \simeq \mathbb{R}^n \times \mathrm{SO}(n)$, thus clearly $\mathbb{R}^n \times \mathrm{Spin}(n)$ is a principal $\mathrm{Spin}(n)$-bundle and the map given by $Id \times \mu : \mathbb{R}^n \times \mathrm{Spin}(n) \to \mathbb{R}^n \times \mathrm{SO}(n)$ is an equivariant 2-fold cover. Thus we get a spin structure on $\mathbb{R}^n$ and since $\mathbb{R}^n$ is contractible this is the only such structure.

Now, if Γ is a Bieberbach group we have a left action of Γ on $\mathrm{B}(\mathbb{R}^n)$ given by $\gamma \cdot (x, (w_1,\dots,w_n)) = (\gamma x, (\gamma_* w_1, \dots, \gamma_* w_n))$. If $\gamma = BL_b$ then $\gamma_* w_j = w_j B$. Fix $(v_1,\dots,v_n) \in \mathrm{B}(\mathbb{R}^n)$. Since $(w_1,\dots,w_n) = (v_1 g,\dots,v_n g)$ for some $g \in \mathrm{SO}(n)$, we see that $\gamma_* w_j = (v_j g)B = v_j(Bg)$, thus the action of Γ on $\mathrm{B}(\mathbb{R}^n)$ corresponds to the action of Γ on $\mathbb{R}^n \times \mathrm{SO}(n)$ given by $\gamma \cdot (x, g) = (\gamma x, Bg)$.

Now assume a group homomorphism is given

$$
\varepsilon : \Gamma \to \mathrm{Spin}(n) \quad \text{ such that } \quad \mu \circ \varepsilon = r, \tag{1.6}
$$

where $r(\gamma) = B$ if $\gamma = BL_b \in \Gamma$. In this case we can lift the left action of Γ on $\mathrm{B}(\mathbb{R}^n)$ to $\tilde{\mathrm{B}}(\mathbb{R}^n) \simeq \mathbb{R}^n \times \mathrm{Spin}(n)$ via $\gamma \cdot (x, \tilde g) = (\gamma x, \varepsilon(\gamma)\tilde g)$. Thus we

have the spin structure

$$
\begin{tikzcd}
\Gamma\backslash(\mathbb{R}^n \times \mathrm{Spin}(n)) \arrow[rr, "\overline{Id\times\mu}"] \arrow[dr] & & \Gamma\backslash(\mathbb{R}^n \times \mathrm{SO}(n)) \arrow[dl] \\
& \Gamma\backslash\mathbb{R}^n &
\end{tikzcd}
\tag{1.7}
$$

for M_Γ since $\Gamma\backslash \mathrm{B}(\mathbb{R}^n) \simeq \mathrm{B}(\Gamma\backslash\mathbb{R}^n)$ and $\overline{Id \times \mu}$ is equivariant. In this way, for each such homomorphism ε, we obtain a spin structure on M_Γ. Furthermore, all spin structures on M_Γ are obtained in this manner (see [**Fr2**], [**LM**]). Throughout the paper we shall denote by (M_Γ, ε) a spin Bieberbach manifold endowed with the spin structure (1.7) induced by ε as in (1.6).

Definition 1.1. Since $r(L_\lambda) = Id$ for $\lambda \in \Lambda$, then $\varepsilon(\lambda) = \pm 1$ for any $\lambda \in \Lambda$. Denote by $\delta_\varepsilon := \varepsilon_{|\Lambda}$, the character of Λ induced by ε. We will say that a spin structure ε on a flat manifold M_Γ is of *trivial type* if $\delta_\varepsilon \equiv 1$. For a torus T_Λ, the only such structure is the trivial structure corresponding to $\varepsilon \equiv 1$.

Remark 1.2. The n-torus admits 2^n spin structures ([**Fr**]). An arbitrary flat manifold M_Γ need not admit any. In [**MP**], we give necessary and sufficient conditions for existence in the case when Γ has holonomy group $\mathbb{Z}_2^k$ and several simple examples of flat manifolds that can not carry a spin structure. Also, we exhibit pairs of manifolds, isospectral on p-forms for all $0 \le p \le n$, where one carries several spin (or pin$^\pm$) structures and the other admits none.

Twisted spinor bundles. Let (L, S) be the spin representation of $\mathrm{Spin}(n)$. Then $\dim(\mathrm{S}) = 2^m$, with $m = [\frac{n}{2}]$. If n is odd then L is irreducible, while if n is even then $\mathrm{S} = \mathrm{S}^+ \oplus \mathrm{S}^-$, where $\mathrm{S}^\pm$ are invariant irreducible subspaces of dimension 2^{m-1} (see Appendix).

The complex flat vector bundles over M_Γ are in a one to one correspondence with complex unitary representations $\rho : \Gamma \to \mathrm{U}(V)$. For simplicity, in this paper we shall only consider representations ρ of Γ such that $\rho_{|\Lambda} = 1$. The group $\mathrm{Spin}(n)$ acts on the right on $\tilde{\mathrm{B}}(\mathbb{R}^n) \times \mathrm{S} \otimes V$ by

$$(b, w \otimes v) \cdot \tilde{g} = (b\tilde{g}, L(\tilde{g}^{-1})(w) \otimes v)$$

and this action defines an equivalence relation such that

$$((x, \tilde{g}), w \otimes v) \sim ((x, 1), L(\tilde{g})(w) \otimes v)$$

for $x \in \mathbb{R}^n$, $\tilde{g} \in \mathrm{Spin}(n)$, $w \otimes v \in \mathrm{S} \otimes V$.

There is a bundle map from the associated bundle $\tilde{\mathrm{B}}(\mathbb{R}^n) \times_{L\otimes Id} \mathrm{S} \otimes V$ onto $\mathbb{R}^n \times (\mathrm{S} \otimes V)$, given by $\overline{((x, \tilde{g}), w \otimes v)} \mapsto (x, L(\tilde{g})w \otimes v)$, which is clearly an isomorphism. Given ρ as above, since $\gamma \cdot \overline{((x, \tilde{g}), w \otimes v)} = \overline{((\gamma x, \varepsilon(\gamma)\tilde{g}), w \otimes \rho(\gamma)v)}$, then the corresponding action of $\gamma = BL_b \in \Gamma$ on $\mathbb{R}^n \times (\mathrm{S} \otimes V)$ is given by

$$\gamma \cdot (x, w \otimes v) = \big(\gamma x, L(\varepsilon(\gamma))(w) \otimes \rho(\gamma)(v)\big). \tag{1.8}$$

In this way we get that the bundle $\Gamma\backslash(\tilde{\mathrm{B}}(\mathbb{R}^n)\times_{L\otimes Id}(\mathrm{S}\otimes V))\to\Gamma\backslash\mathbb{R}^n=M_\Gamma$, defined by ε, is isomorphic to

$$S_\rho(M_\Gamma,\varepsilon):=\Gamma\backslash(\mathbb{R}^n\times(\mathrm{S}\otimes V))\to\Gamma\backslash\mathbb{R}^n=M_\Gamma \tag{1.9}$$

called the *spinor bundle of* M_Γ *with twist* V.

Now denote by $\Gamma^\infty(S_\rho(M_\Gamma,\varepsilon))$ the space of smooth sections of $S_\rho(M_\Gamma,\varepsilon)$, i.e. the space of *spinor fields* of M_Γ. Let $\psi:\mathbb{R}^n\to\mathbb{R}^n\times(\mathrm{S}\otimes V)$ be given by $\psi(x)=(x,f(x))$ where $f:\mathbb{R}^n\to\mathrm{S}\otimes V$ is smooth. We have that ψ defines a section of $\Gamma\backslash(\mathbb{R}^n\times(\mathrm{S}\otimes V))$ if and only if, for each $\gamma\in\Gamma$, $\psi(\gamma x)\sim\psi(x)$, that is, if and only if $\psi(\gamma x)=\tilde\gamma\psi(x)$, for some $\tilde\gamma\in\Gamma$. Since Γ acts freely on $\mathbb{R}^n$, this is the case if and only if $\tilde\gamma=\gamma$ and furthermore f satisfies $f(\gamma x)=(L\circ\varepsilon\otimes\rho)(\gamma)f(x)$. In other words, $\Gamma^\infty(S_\rho(M_\Gamma,\varepsilon))$ can be identified to the space:

$$\{f:\mathbb{R}^n\to\mathrm{S}\otimes V\text{ smooth}\,:\,f(\gamma x)=(L\circ\varepsilon\otimes\rho)(\gamma)f(x)\}.$$

In particular, in the case of $T_\Lambda=\Lambda\backslash\mathbb{R}^n$, if $\psi(x)=(x,f(x))$ is a section of $\Lambda\backslash(\mathbb{R}^n\times(\mathrm{S}\otimes V))$ then, since we have assumed that $\rho_{|\Lambda}=Id$, in the notation of Definition 1.1,

$$f(x+\lambda)=(L\circ\varepsilon\otimes\rho)(\lambda)f(x)=\delta_\varepsilon(\lambda)f(x). \tag{1.10}$$

Thus f is δ_ε-equivariant. Conversely, if $f:\mathbb{R}^n\to\mathrm{S}\otimes V$ is δ_ε-equivariant then $\psi(x)=(x,f(x))$ is a spinor field on T_Λ.

Now $\delta_\varepsilon\in\mathrm{Hom}(\Lambda,\{\pm1\})$, hence there exists $u_\varepsilon\in\mathbb{R}^n$ such that $\delta_\varepsilon(\lambda)=e^{2\pi i\,u_\varepsilon\cdot\lambda}$ for all $\lambda\in\Lambda$. Set

$$\Lambda_\varepsilon^*:=\Lambda^*+u_\varepsilon \tag{1.11}$$

where Λ^* is the dual lattice of Λ. If $\lambda_1,\dots,\lambda_n$ is a $\mathbb{Z}$-basis of Λ, let $\lambda_1',\dots,\lambda_n'$ be the dual basis and set

$$J_\varepsilon^\pm:=\{i\in\{1,\dots,n\}:\varepsilon(L_{\lambda_i})=\delta_\varepsilon(\lambda_i)=\pm1\}. \tag{1.12}$$

We thus have,

$$u_\varepsilon=\tfrac12\sum_{i\in J_\varepsilon^-}\lambda_i'\quad\mod\Lambda^*, \tag{1.13}$$

and furthermore

$$\Lambda_\varepsilon^*=\bigoplus_{j\in J_\varepsilon^+}\mathbb{Z}\lambda_j'\oplus\bigoplus_{j\in J_\varepsilon^-}(\mathbb{Z}+\tfrac12)\lambda_j'. \tag{1.14}$$

For $u\in\Lambda_\varepsilon^*$, $w\in\mathrm{S}\otimes V$ consider the function

$$f_{u,w}(x):=f_u(x)w:=e^{2\pi i\,u\cdot x}w. \tag{1.15}$$

It is clear that $f_{u,w}$ is δ_ε-equivariant and hence $\psi_{u,w}(x):=(x,f_{u,w}(x))$ gives a spinor field on T_Λ.

2. The spectrum of twisted Dirac operators.

In this section we will introduce the Dirac operator D_ρ acting on sections of the spinor bundle $S_\rho(M_\Gamma, \varepsilon)$ (see (1.9)), where (ρ, V) is a finite dimensional, unitary representation of Γ. We shall denote by χ_ρ and d_ρ the character and the dimension of ρ, respectively. In the main results in this section we will give an explicit formula for the multiplicities of the eigenvalues of twisted Dirac operators of any spin Bieberbach manifold (M_Γ, ε) together with a general expression for the η-series. We shall later use this expression to compute the η-invariant for flat manifolds with holonomy group $\mathbb{Z}_2^k$ (see (3.9)) and for certain p-dimensional flat manifolds with holonomy group $\mathbb{Z}_p$, p prime (see (5.2)).

The spectrum of twisted Dirac operators. If M_Γ is a flat manifold endowed with a spin structure ε, let $\Delta_{s,\rho}$ denote the *twisted spinor Laplacian* acting on smooth sections of the spinor bundle $S_\rho(M_\Gamma, \varepsilon)$. That is, if $\psi(x) = (x, f(x))$ is a spinor field with $f(x) = \sum_{i=1}^d f_i(x) w_i$, $f_i : \mathbb{R}^n \to \mathbb{C}$ smooth and $\{w_i : 1 \le i \le d\}$ a basis of $\mathrm{S} \otimes V$ $(d = 2^m d_\rho)$ then

$$\Delta_{s,\rho}\psi(x) = \Big(x, \sum_{i=1}^{d} \Delta f_i(x) w_i\Big), \tag{2.1}$$

where Δ is the Laplacian on functions on M_Γ.

It is easy to see that for $u \in \Lambda_\varepsilon^*$, $w \in \mathrm{S} \otimes V$, every $f_{u,w}$ as in (1.15) is an eigenfunction of $\Delta_{s,\rho}$ with eigenvalue $-4\pi^2\|u\|^2$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $\mathbb{R}^n$ and let ψ be as above. The *twisted Dirac operator* D_ρ is defined by

$$D_\rho\psi(x) = \Big(x, \sum_{i=1}^{n} e_i \,\frac{\partial f}{\partial x_i}\,(x)\Big) \tag{2.2}$$

where e_i acts by $L(e_i) \otimes Id$ on $\mathrm{S} \otimes V$. We will often abuse notation and assume that D_ρ acts on the function f where $\psi(x) = (x, f(x))$, writing $D_\rho f(x) = \sum_{i=1}^n e_i \frac{\partial}{\partial x_i} f(x)$.

One has that D_ρ is an elliptic, first order, essentially self-adjoint differential operator on the spinor bundle $S_\rho(M_\Gamma, \varepsilon)$ of M_Γ. Furthermore, D_ρ has a discrete spectrum consisting of real eigenvalues of finite multiplicity and satisfies $D_\rho^2 = -\Delta_{s,\rho}$. We will denote by $Spec_D(M_\Gamma, \varepsilon)$ the spectrum of D_ρ when ρ is understood.

We now compute the action of D_ρ on $f_{u,w}$ for $u \in \Lambda_\varepsilon^*$, $w \in \mathrm{S} \otimes V$. We have

$$D_\rho f_{u,w}(x) = \sum_{j=1}^{n} e_j \cdot \frac{\partial}{\partial x_j} e^{2\pi i u\cdot x} w = 2\pi i\, e^{2\pi i u\cdot x} u \cdot w = 2\pi i u \cdot f_{u,w}(x).$$

For any $u \in \mathbb{R}^n \smallsetminus \{0\}$, left Clifford multiplication by u on S is given by $u\cdot w = L(u)(w)$ for $w \in \mathrm{S}$. We fix $\langle\, ,\, \rangle$ an inner product on S such that $L(u)$ is

skew Hermitian for every $u \in \mathbb{R}^n \smallsetminus \{0\}$, hence $\langle\, , \,\rangle$ is Spin(n)-invariant. Note that $L(u)^2 = -\|u\|^2 Id$. For each $u \in \Lambda_\varepsilon^*$ with $\|u\| = \mu > 0$, let $\{w_j^\pm\}_{j=1}^{2^{m-1}}$ be an orthonormal basis of the eigenspace of $L(u)$ with eigenvalue $\mp i\|u\|$. Let $\{v_k\}_{k=1}^{d_\rho}$ be an orthonormal basis of V. If we set

$$f_{u,j,k}^\pm(x) := f_{u,w_j^\pm \otimes v_k}(x) = e^{2\pi i u\cdot x} w_j^\pm \otimes v_k, \tag{2.3}$$

for $u \in \Lambda_\varepsilon^*$, $1 \le j \le 2^{m-1}$ and $1 \le k \le d_\rho$, then we have

$$D_\rho f_{u,j,k}^\pm = \pm 2\pi \|u\|\, f_{u,j,k}^\pm,$$

that is, $f_{u,j,k}^\pm \in H_\mu^\pm$, the space of eigensections of D_ρ with eigenvalue $\pm 2\pi\mu$.

If $u = 0$, $w \ne 0$, by (1.10) the constant function $f_{0,w}(x) = w$ with $w \in \mathrm{S} \otimes V$ defines a spinor field if and only if

$$f_{0,w}(x + \lambda) = \delta_\varepsilon(\lambda) f_{0,w}(x)$$

for any $\lambda \in \Lambda$. That is, if and only if $\varepsilon = 1$, the trivial spin structure. Moreover, $f_{0,w}$ is an eigenfunction of D_ρ with eigenvalue 0, i.e. a *harmonic spinor*. Conversely, if $D_\rho f = 0$ then also $\Delta_{s,\rho} f = 0$ hence, it follows from (2.1) that $f = f_{0,w}$, a constant function, for some $w \in \mathrm{S} \otimes V$.

We shall denote by H_0 the space of harmonic spinors. If n is even, then sections of the form $f_{0,w}$ with $w \in \mathrm{S}^+ \otimes V$ (resp. $\mathrm{S}^- \otimes V$) are often called *positive* (resp. *negative*) harmonic spinors. We have $H_0 = H_0^+ \oplus H_0^-$, where H_0^+ and H_0^- respectively denote the spaces of positive and negative harmonic spinors.

If $\mu > 0$ we set $d_{\rho,\mu}^\pm(\Lambda, \varepsilon) := \dim H_\mu^\pm$, the multiplicities of the eigenvalues $\pm 2\pi\mu$. If $\mu = 0$, let $d_{\rho,0}(\Lambda, \varepsilon) := \dim H_0$ and $d_{\rho,0}^\pm(\Lambda, \varepsilon) := \dim H_0^\pm$, if n is even.

The next result gives $Spec_D(T_\Lambda, \varepsilon)$ for the torus T_Λ and shows that it is determined by the cardinality of the sets

$$\Lambda_{\varepsilon,\mu}^* := \{v \in \Lambda_\varepsilon^* \;:\; \|v\| = \mu\}. \tag{2.4}$$

where Λ_ε^* is as in (1.11).

Proposition 2.1. *Let ε be a spin structure on the torus $T_\Lambda \simeq \Lambda\backslash\mathbb{R}^n$ and let $m = [\frac{n}{2}]$. In the notation above, we have:*

(i) $H_0 = S \otimes V$ if $\varepsilon = 1$ and $H_0 = 0$ if $\varepsilon \ne 1$.

(ii) If $\mu > 0$, then

$$H_\mu^\pm = span\{f_{u,j,k}^\pm : u \in \Lambda_{\varepsilon,\mu}^*,\ 1 \le j \le 2^{m-1},\ 1 \le k \le d_\rho\}$$

with $f_{u,j,k}^\pm$ as in (2.3). The multiplicity of the eigenvalue $\pm 2\pi\mu$ of D_ρ thus equals

$$d_{\rho,\mu}^\pm(\Lambda, \varepsilon) = 2^{m-1} d_\rho |\Lambda_{\varepsilon,\mu}^*|$$

with $\Lambda_{\varepsilon,\mu}^*$ as in (2.4). Furthermore, $\|f_{u,j,k}^\pm\| = vol(T_\Lambda)^{1/2}$ for each u, j, k.

Proof. The statements in (i) are clear in light of the discussion before the proposition. Now let $L^2(\Lambda\backslash\mathbb{R}^n;\delta_\varepsilon)$ denote the space

$$\Big\{f:\mathbb{R}^n\to\mathbb{C}\mid f(x+\lambda)=\delta_\varepsilon(\lambda)f(x)\text{ for }\lambda\in\Lambda,x\in\mathbb{R}^n\text{ and }\int_{T_\Lambda}|f|^2<\infty\Big\}.$$

and let $L^2(\Lambda\backslash\mathbb{R}^n,\mathrm{S}\otimes V;\delta_\varepsilon)$ be defined similarly by using functions with values in $\mathrm{S}\otimes V$.

For each $u\in\Lambda^*_\varepsilon$ the function $f_u(x)=e^{2\pi iu\cdot x}$ lies in $L^2(\Lambda\backslash\mathbb{R}^n;\delta_\varepsilon)$. In the case $\varepsilon=1$, the Stone-Weierstrass theorem implies that the set $\{f_u:u\in\Lambda^*\}$ is a complete orthogonal system of $L^2(\Lambda\backslash\mathbb{R}^n)$. Since $\Lambda^*_\varepsilon=\Lambda^*+u_\varepsilon$, this implies that the set $\{f_u:u\in\Lambda^*_\varepsilon\}$ is a complete orthogonal system of $L^2(\Lambda\backslash\mathbb{R}^n;\delta_\varepsilon)$. Now, if for each given $u\in\Lambda^*_\varepsilon$, we choose an orthonormal basis $\mathcal{B}_u$ of $\mathrm{S}\otimes V$ of eigenvectors of $L(u)\otimes Id$, then this clearly implies that the set

$$\{f_{u,w}(x):u\in\Lambda^*_\varepsilon,w\in\mathcal{B}_u\}\tag{2.5}$$

is a complete orthogonal system of $L^2(\Lambda\backslash\mathbb{R}^n,\mathrm{S}\otimes V;\delta_\varepsilon)$. Furthermore each $f_{u,w}$ is an eigenfunction of D_ρ with eigenvalue $\pm2\pi\|u\|$, therefore (ii) follows. □

Remark 2.2. If Λ is the canonical (or cubic) lattice, then

$$|\Lambda^*_{\varepsilon,\mu}|=|\{(m_1,\dots,m_n)\in\mathbb{Z}^n:\sum_{j\in J_\varepsilon^+}m_j^2+\sum_{j\in J_\varepsilon^-}(m_j+\tfrac12)^2=\mu^2\}|,\tag{2.6}$$

where $J_\varepsilon^\pm$ (see (1.12)) are computed with respect to the canonical basis of $\mathbb{R}^n$. We observe that $|J_\varepsilon^-|$ (or $|J_\varepsilon^+|$, see (1.14)) determines the multiplicity of the eigenvalue $\pm2\pi\mu$, for any $\mu>0$, by (2.6) and by the multiplicity formula in (ii) of Proposition 2.1. The converse also holds, thus we have that $Spec_D(T_\Lambda,\varepsilon)=Spec_D(T_\Lambda,\varepsilon')$ if and only if $|J_\varepsilon^-|=|J_{\varepsilon'}^-|$.

Our next goal is to obtain a formula for the multiplicities of the eigenvalues $\pm2\pi\mu$, $\mu\geq0$, of the Dirac operator acting on twisted spinor fields of an arbitrary spin flat manifold (M_Γ,ε). We shall see that only the elements in Γ such that $B\in F_1$ (see 1.2) will contribute to the formula. One of the key ingredients in the formula will be a sign, attached to each pair γ,u, with $\gamma=BL_b$, $u\in\Lambda^*$ fixed by B. This sign appears when comparing the conjugacy classes of two elements x,y in $\mathrm{Spin}(n-1)$ that are conjugate in $\mathrm{Spin}(n)$. By Lemma 6.2, given two such elements x,y, then either y is conjugate to x or to $-e_1xe_1$ in $\mathrm{Spin}(n-1)$ (see Appendix). In what follows we shall write $x\sim y$ if x,y are conjugate in $\mathrm{Spin}(n-1)$.

If $\gamma=BL_b\in\Lambda\backslash\Gamma$ and $u\in(\Lambda^*_\varepsilon)^B\smallsetminus\{0\}$, let $h_u\in\mathrm{Spin}(n)$ be such that $h_u\,u\,h_u^{-1}=\|u\|e_n$. Therefore, since $Bu=u$, $h_u\varepsilon(\gamma)h_u^{-1}\in\mathrm{Spin}(n-1)$, by the comments after Definition 6.5.

We shall make use of the following definitions.

Definition 2.3. Fix an element x_γ in the maximal torus T of $\mathrm{Spin}(n-1)$ (see (1.5)) that is conjugate in $\mathrm{Spin}(n)$ to $\varepsilon(\gamma)$. Define $\sigma_\varepsilon(u,x_\gamma)=1$ if

$h_u \varepsilon(\gamma) h_u^{-1}$ is conjugate to x_γ in $\mathrm{Spin}(n-1)$ and $\sigma_\varepsilon(u, x_\gamma) = -1$, otherwise (in this case $h_u \varepsilon(\gamma) h_u^{-1} \sim -e_1 x_\gamma e_1$).

Note that $\sigma_\varepsilon(u, x_\gamma)$ is independent of the choice of h_u. For simplicity, we shall simply write $\sigma(u, x_\gamma)$ when ε is understood.

Remark 2.4. If n is even, then $\sigma(u, x_\gamma) = 1$ for all u, by Lemma 6.2. Also, if γ is such that $n_B > 1$ then, by arguing as at the end of the proof of Lemma 6.2, we see that $\sigma(u, x_\gamma) = 1$ for all u. Furthermore, if x_γ is not conjugate to $-e_1\, x_\gamma\, e_1$ in $\mathrm{Spin}(n)$, then, by the definition, $\sigma(u, -e_1\, x_\gamma\, e_1) = -\sigma(u, x_\gamma)$. Moreover $\sigma(-u, x_\gamma) = -\sigma(u, x_\gamma)$ and $\sigma(\alpha u, x_\gamma) = \sigma(u, x_\gamma)$ for any $\alpha > 0$, since we may take $h_{-u} = e_1 h_u$ and $h_{\alpha u} = h_u$, respectively.

Now, let as usual χ_ρ, χ_L and $\chi_{L^\pm}$ denote the characters of ρ, L and $L^\pm$, respectively. Keeping the notation above, for each $\gamma = BL_b \in \Gamma$, $\mu > 0$ we set

$$e_{\mu,\gamma,\sigma}(\delta_\varepsilon) := \sum_{u \in (\Lambda^*_{\varepsilon,\mu})^B} \sigma(u, x_\gamma) e^{-2\pi i u \cdot b} \tag{2.7}$$

where $(\Lambda^*_{\varepsilon,\mu})^B = \{v \in \Lambda^*_{\varepsilon,\mu} : Bv = v\}$. When $\sigma = 1$ we just write $e_{\mu,\gamma}(\delta_\varepsilon)$ for $e_{\mu,\gamma,1}(\delta_\varepsilon)$.

We are now in a position to prove the main result in this section.

Theorem 2.5. *Let Γ be a Bieberbach group with translation lattice Λ and holonomy group $F \simeq \Lambda\backslash\Gamma$. Assume $M_\Gamma = \Gamma\backslash\mathbb{R}^n$ is a spin compact flat manifold, with spin structure ε. Then, if n is even, for each $\mu > 0$ the multiplicity of the eigenvalue $\pm 2\pi\mu$ of D_ρ is given by*

$$d^\pm_{\rho,\mu}(\Gamma, \varepsilon) = \tfrac{1}{|F|} \sum_{\gamma \in \Lambda\backslash\Gamma} \chi_\rho(\gamma)\, e_{\mu,\gamma}(\delta_\varepsilon)\, \chi_{L_{n-1}}(x_\gamma). \tag{2.8}$$

If n is odd then

$$\begin{aligned} d^\pm_{\rho,\mu}(\Gamma, \varepsilon) \;=\; & \tfrac{1}{|F|} \Big(\sum_{\substack{\gamma \in \Lambda\backslash\Gamma \\ B \notin F_1}} \chi_\rho(\gamma)\, e_{\mu,\gamma}(\delta_\varepsilon)\, \chi_{L^\pm_{n-1}}(x_\gamma) \,+ \\ & \sum_{\substack{\gamma \in \Lambda\backslash\Gamma \\ B \in F_1}} \chi_\rho(\gamma) \sum_{u \in (\Lambda^*_{\varepsilon,\mu})^B} e^{-2\pi i u \cdot b}\, \chi_{L^{\pm\sigma(u,x_\gamma)}_{n-1}}(x_\gamma) \Big) \end{aligned} \tag{2.9}$$

with $e_{\mu,\gamma}(\delta_\varepsilon)$ as in (2.7), F_1 as in (1.2), and x_γ, $\sigma(u, x_\gamma)$ as in Definition 2.3.

Let $\mu = 0$. If $\varepsilon_{|\Lambda} \neq 1$ then $d_{\rho,0}(\Gamma, \varepsilon) = 0$. If $\varepsilon_{|\Lambda} = 1$ then

$$d_{\rho,0}(\Gamma, \varepsilon) = \tfrac{1}{|F|} \sum_{\gamma \in \Lambda\backslash\Gamma} \chi_\rho(\gamma)\, \chi_{L_n}(\varepsilon(\gamma)) = \dim(S \otimes V)^F. \tag{2.10}$$

We note that the summands in (2.8) and in (2.9) are independent of the representative γ mod Λ and of the choice of x_γ, but in general the individual factors are not.

Proof. We proceed initially as in [**MR3**], [**MR4**]. We have that

$$L^2(S_\rho(M_\Gamma,\varepsilon)) \simeq L^2(S_\rho(T_\Lambda,\varepsilon))^\Gamma = \bigoplus_{\mu>0}\Big((H_\mu^+)^\Gamma \oplus (H_\mu^-)^\Gamma\Big) \oplus {H_0}^\Gamma.$$

Thus, $d^{\pm}_{\rho,\mu}(\Gamma,\varepsilon) = \dim (H_\mu^\pm)^\Gamma$, for $\mu > 0$, and $d_{\rho,0}(\Gamma,\varepsilon) = \dim {H_0}^\Gamma$. One has a projection $p_\mu^\pm$ from $H_\mu^\pm$ onto $(H_\mu^\pm)^\Gamma$:

$$p_\mu^\pm = \tfrac{1}{|\Lambda\backslash\Gamma|}\sum_{\gamma\in\Lambda\backslash\Gamma}\gamma_{|H_\mu^\pm}$$

hence

$$\dim (H_\mu^\pm)^\Gamma = \operatorname{tr} p_\mu^\pm = \tfrac{1}{|\Lambda\backslash\Gamma|}\sum_{\gamma\in\Lambda\backslash\Gamma}\operatorname{tr}(\gamma_{|H_\mu^\pm})$$

and similarly for $\dim {H_0}^\Gamma$, with H_0 in place of $H_\mu^\pm$.

Thus, by (ii) of Proposition 2.1 we have to compute, for $\gamma\in\Gamma$,

$$\operatorname{tr}\gamma_{|H_\mu^\pm} = \frac{1}{\operatorname{vol}(T_\Lambda)}\sum_{u\in\Lambda^*_{\varepsilon,\mu}}\sum_{j=1}^{2^{m-1}}\sum_{k=1}^{d_\rho}\langle \gamma\cdot f^\pm_{u,j,k}, f^\pm_{u,j,k}\rangle. \tag{2.11}$$

Recall, from (1.8), that $\gamma\in\Gamma$ acts by $\gamma\cdot\psi(x) = (\gamma x, (L\circ\varepsilon\otimes\rho)(\gamma)f(x))$ on $\psi(x) = (x, f(x))$. Thus there is an action of γ on f given by

$$\gamma\cdot f(x) = (L\circ\varepsilon\otimes\rho)(\gamma)f(\gamma^{-1}x).$$

Since $\gamma^{-1} = L_{-b}B^{-1}$ by (2.3) we have:

$$\begin{aligned}\gamma\cdot f^\pm_{u,j,k}(x) &= (L\circ\varepsilon\otimes\rho)(\gamma)\, f^\pm_{u,j,k}(\gamma^{-1}x)\\ &= e^{-2\pi i u\cdot b}\, f_{Bu}(x)\, L(\varepsilon(\gamma))w_j^\pm\otimes\rho(\gamma)v_k.\end{aligned} \tag{2.12}$$

Let $\gamma^\pm_{u,j,k} := \langle\gamma\cdot f^\pm_{u,j,k}, f^\pm_{u,j,k}\rangle = \int_{T_\Lambda}\langle\gamma\cdot f^\pm_{u,j,k}(x), f^\pm_{u,j,k}(x)\rangle\, dx$. Now, using (2.12) we compute:

$$\begin{aligned}\gamma^\pm_{u,j,k} &= e^{-2\pi i u\cdot b}\int_{T_\Lambda}\langle\, f_{Bu}(x)L(\varepsilon(\gamma))w_j^\pm\otimes\rho(\gamma)v_k\,,\, f_u(x)w_j^\pm\otimes v_k\,\rangle\, dx\\ &= e^{-2\pi i u\cdot b}\langle\, L(\varepsilon(\gamma))w_j^\pm\otimes\rho(\gamma)v_k\,,\, w_j^\pm\otimes v_k\,\rangle\int_{T_\Lambda}e^{2\pi i(Bu-u)\cdot x}\, dx\\ &= e^{-2\pi i u\cdot b}\langle\, L(\varepsilon(\gamma))\, w_j^\pm\otimes\rho(\gamma)v_k\,,\, w_j^\pm\otimes v_k\,\rangle\operatorname{vol}(T_\Lambda)\,\delta_{Bu,u}.\end{aligned}$$

In this way we get

$$\operatorname{tr} p_\mu^\pm = \frac{1}{|F|}\sum_{\gamma\in\Lambda\backslash\Gamma}\ \sum_{u\,\in\,(\Lambda^*_{\varepsilon,\mu})^B}\ \sum_{k=1}^{d_\rho}\langle\rho(\gamma)v_k, v_k\rangle\sum_{j=1}^{2^{m-1}}e^{-2\pi i u\cdot b}\,\langle L(\varepsilon(\gamma))w_j^\pm, w_j^\pm\rangle.$$

Now, if $\gamma = BL_b\in\Gamma$ and $u\in(\Lambda^*_{\varepsilon,\mu})^B$, then $\varepsilon(\gamma)\in\operatorname{Spin}(n-1,u)$ (see (6.6)). Hence $L(\varepsilon(\gamma))$ preserves the eigenspaces $\mathrm{S}_u^\pm$ of $L(u)\otimes Id$ and we can

consider the trace of $(L\circ\varepsilon\otimes\rho)(\gamma)$ restricted to $\mathrm{S}_u^{\pm}\otimes V$. Thus we finally obtain

$$d^{\pm}_{\rho,\mu}(\Gamma,\varepsilon)=\tfrac{1}{|F|}\sum_{\gamma\in\Lambda\backslash\Gamma}\operatorname{tr}\rho(\gamma)\sum_{u\,\in\,(\Lambda^*_{\varepsilon,\mu})^B} e^{-2\pi i u\cdot b}\operatorname{tr}L(\varepsilon(\gamma))_{|\mathrm{S}_u^{\pm}}. \tag{2.13}$$

The next task will be to compute the traces $\operatorname{tr}L(\varepsilon(\gamma))_{|\mathrm{S}_u^{\pm}}$, showing they can be expressed as values of characters of spin representations. The influence of u will only appear in the determination of a sign. We shall use the element x_γ and the notions introduced in Definition 2.3.

We first note that $\operatorname{tr}L(\varepsilon(\gamma))_{|S_u^{\pm}}=\operatorname{tr}L(h_u\varepsilon(\gamma)h_u^{-1})_{|S_{e_n}^{\pm}}$. Now, we use Lemma 6.6 together with Definition 2.3. If n odd and $h_u\varepsilon(\gamma)h_u^{-1}\sim x_\gamma$ then $\operatorname{tr}L(h_u\varepsilon(\gamma)h_u^{-1})_{|S_{e_n}^{\pm}}=\operatorname{tr}L^{\pm}_{n-1}(x_\gamma)$. If $h_u\varepsilon(\gamma)h_u^{-1}\not\sim x_\gamma$, then $h_u\varepsilon(\gamma)h_u^{-1}\sim -e_1x_\gamma e_1$, hence

$$\operatorname{tr}L(h_u\varepsilon(\gamma)h_u^{-1})_{|S_{e_n}^{\pm}}=\operatorname{tr}L^{\pm}_{n-1}(-e_1x_\gamma e_1)=\operatorname{tr}L^{\mp}_{n-1}(x_\gamma)$$

since $L(e_1)$ sends $\mathrm{S}^{\pm}$ to $\mathrm{S}^{\mp}$ orthogonally. For n even we proceed similarly, using (6.3). Thus we obtain:

$$\operatorname{tr}L(\varepsilon(\gamma))_{|S_u^{\pm}}=\begin{cases}\operatorname{tr}L_{n-1}(x_\gamma) & n \text{ even}\\ \operatorname{tr}L_{n-1}^{\pm\sigma(u,x_\gamma)}(x_\gamma) & n \text{ odd.}\end{cases} \tag{2.14}$$

Substituting (2.14) in (2.13) we get that $d^{\pm}_{\rho,\mu}(\Gamma,\varepsilon)$ equals

$$\tfrac{1}{|F|}\sum_{\gamma\in\Lambda\backslash\Gamma}\chi_\rho(\gamma)\sum_{u\,\in\,(\Lambda^*_{\varepsilon,\mu})^B} e^{-2\pi i u\cdot b}\begin{cases}\operatorname{tr}L_{n-1}(x_\gamma) & n \text{ even}\\ \operatorname{tr}L_{n-1}^{\pm\sigma(u,x_\gamma)}(x_\gamma) & n \text{ odd}\end{cases}$$

as asserted in formula (2.8) for n even. If n is odd, then, by separating the contributions of the elements $\gamma=BL_b$ with $B\in F_1$ from those with $B\notin F_1$ (see Remark 2.4), we arrive at formula (2.9).

In the case when $\mu=0$ we may proceed in a similar way. If we identify $w\otimes v\in\mathrm{S}\otimes V$ with the constant function $f_{0,w\otimes v}$, then for $\gamma\in\Gamma$ we have

$$\operatorname{tr}\gamma_{|H_0}=\tfrac{1}{\operatorname{vol}T_\Lambda}\sum_{j=1}^{2^m}\sum_{k=1}^{d_\rho}\langle\gamma\cdot w_j\otimes v_k, w_j\otimes v_k\rangle=\chi_\rho(\gamma)\;\chi_{L_n}(\varepsilon(\gamma)).$$

Thus

$$\dim(H_0^\Gamma)=\tfrac{1}{|F|}\sum_{\gamma\in\Lambda\backslash\Gamma}\operatorname{tr}\gamma_{|H_0}=\tfrac{1}{|F|}\sum_{\gamma\in\Lambda\backslash\Gamma}\chi_\rho(\gamma)\;\chi_{L_n}(\varepsilon(\gamma))$$

as claimed. Concerning the last equality in the theorem, we know that

$$H_0=\{f_{0,w\otimes v}: L\circ\varepsilon\otimes\rho(\gamma)w\otimes v=w\otimes v\}\simeq(\mathrm{S}\otimes V)^F,$$

since $F\simeq\Lambda\backslash\Gamma$ and Λ acts trivially for ε of trivial type. □

Corollary 2.6. *Let (M_Γ,ε) be a spin compact flat manifold of dimension n. If n is even, or if n odd and $n_B>1$ for every $\gamma=BL_b\in\Gamma$ (i.e. $F_1=\emptyset$), then the spectrum of the twisted Dirac operator D_ρ is symmetric.*

Proof. If n is even, the assertion is automatic by (2.8). If n is odd, by (2.9) and (6.4) we have that $d^+_{\rho,\mu}(\Gamma,\varepsilon) - d^-_{\rho,\mu}(\Gamma,\varepsilon)$ equals

$$\frac{(2i)^m}{|F|} \sum_{\gamma\in\Lambda\backslash\Gamma} \chi_\rho(\gamma)\, e_{\mu,\gamma}(\delta_\varepsilon) \prod_{j=1}^{m} \sin t_j(x_\gamma) \tag{2.15}$$

where $x_\gamma = x(t_1(x_\gamma), \ldots, t_m(x_\gamma))$, in the notation of (1.4). It is clear that γ satisfies $n_B > 1$ if and only if $t_k(x_\gamma) \in \pi\mathbb{Z}$, for some k, hence in this case $\prod_{j=1}^m \sin t_j(x_\gamma) = 0$. Thus, if $n_B > 1$ for every γ, (2.15) implies that the Dirac spectrum is symmetric. □

Remark 2.7. In specific examples, the expressions for the multiplicities of eigenvalues in the theorem, can be made more explicit by substituting $\chi_{L_{n-1}}(x_\gamma)$, $\chi_{L^\pm_{n-1}}(x_\gamma)$ or $\chi_{L_n}(\varepsilon(\gamma))$ by the values given in Lemma 6.1 in terms of products of cosines or sines of the $t_j(x_\gamma)$, where $x_\gamma = x(t_1(x_\gamma), \ldots, t_m(x_\gamma))$ as above.

Remark 2.8. Set $d_{\rho,\mu}(\Gamma,\varepsilon) := d^+_{\rho,\mu}(\Gamma,\varepsilon) + d^-_{\rho,\mu}(\Gamma,\varepsilon)$ for $\mu > 0$. Note that, since $D^2_\rho = -\Delta_{s,\rho}$, then $d_{\rho,\mu}(\Gamma,\varepsilon)$ is just the multiplicity of the eigenvalue $4\pi^2\mu^2$ of $-\Delta_{s,\rho}$. We thus have:

$$d_{\rho,\mu}(\Gamma,\varepsilon) = \frac{\alpha(n)}{|F|} \sum_{\gamma\in\Lambda\backslash\Gamma} \chi_\rho(\gamma)\; e_{\mu,\gamma}(\delta_\varepsilon)\; \chi_{L_{n-1}}(\varepsilon(\gamma)) \tag{2.16}$$

where $\alpha(n) = 1$ or 2 depending on whether n is odd or even. Clearly, (twisted) Dirac isospectrality implies (twisted) spinor Laplacian isospectrality, but we shall see the converse is not true (see Example 4.4).

Spectral asymmetry and η-series. We decompose $Spec_D(M_\Gamma,\varepsilon) = \mathcal{S} \,\dot\cup\, \mathcal{A}$ where $\mathcal{S}$ and $\mathcal{A}$ are the symmetric and the asymmetric components of the spectrum, respectively. That is, if $\lambda = 2\pi\mu$, $\lambda \in \mathcal{S}$ if and only if $d^+_{\rho,\mu}(\Gamma,\varepsilon) = d^-_{\rho,\mu}(\Gamma,\varepsilon)$. We say that $Spec_D(M_\Gamma,\varepsilon)$ is symmetric if $\mathcal{A} = \emptyset$. In this case, the positive spectrum $Spec^+_D(M_\Gamma) = \{\lambda \in Spec_D(M_\Gamma) : \lambda > 0\}$ and H_0 determine the whole spectrum $Spec_D(M_\Gamma)$. The symmetry of the spectrum of the Dirac operator depends on Γ and also on the spin structure ε on M_Γ.

Our next goal is to derive an expression for the η-series $\eta_{(\Gamma,\rho,\varepsilon)}(s)$, for a general Bieberbach manifold M_Γ with a spin structure ε. Consider

$$\sum_{\substack{\lambda\,\in\, Spec_D(M_\Gamma,\,\varepsilon)\\ \lambda\neq 0}} \frac{\operatorname{sgn}(\lambda)}{|\lambda|^s} = \frac{1}{(2\pi)^s} \sum_{\mu\in\frac{1}{2\pi}\mathcal{A}} \frac{d^+_{\rho,\mu}(\Gamma,\varepsilon) - d^-_{\rho,\mu}(\Gamma,\varepsilon)}{|\mu|^s}. \tag{2.17}$$

It is known that this series converges absolutely for $Re(s) > n$ and defines a holomorphic function $\eta_{(\Gamma,\rho,\varepsilon)}(s)$ in this region, having a meromorphic continuation to $\mathbb{C}$, with simple poles (possibly) at $z = n - k$, $k \in \mathbb{N}_0$, that is holomorphic at $s = 0$ ([**APS**] and [**Gi**]). One defines the eta-invariant of M_Γ by $\eta_{(\Gamma,\rho,\varepsilon)} := \eta_{(\Gamma,\rho,\varepsilon)}(0)$. By Corollary 2.6, $\eta(s) \equiv 0$ if $n = 2m$ or if

$n = 2m+1$ and $F_1 = \emptyset$. Furthermore, it is known that if $n \not\equiv 3 \mod (4)$ then $\eta(s) \equiv 0$ for every Riemannian manifold M. In what follows we will often write $\eta(s)$ and η in place of $\eta_{(\Gamma,\rho,\varepsilon)}(s)$ and $\eta_{(\Gamma,\rho,\varepsilon)}$, for simplicity.

Proposition 2.9. *Let Γ be a Bieberbach group of dimension $n = 4r+3$ (thus $m = 2r+1$) with holonomy group F and let ε be a spin structure on $M_\Gamma = \Gamma\backslash\mathbb{R}^n$. Then the η-series of M_Γ is given by*

$$\eta_{(\Gamma,\rho,\varepsilon)}(s) = \tfrac{(2i)^m}{|F|(2\pi)^s} \sum_{\substack{\gamma \in \Lambda\backslash\Gamma \\ B \in F_1}} \chi_\rho(\gamma) \Big(\prod_{j=1}^m \sin t_j(x_\gamma)\Big) \sum_{\mu \in \frac{1}{2\pi}\mathcal{A}} \frac{e_{\mu,\gamma,\sigma}(\delta_\varepsilon)}{|\mu|^s} \tag{2.18}$$

where $e_{\mu,\gamma,\sigma}(\delta_\varepsilon) = \sum_{u \in (\Lambda^*_{\varepsilon,\mu})^B} \sigma(u,x_\gamma) e^{-2\pi i u\cdot b}$ as in (2.7) and, in the notation of (1.4), $x_\gamma = x(t_1(x_\gamma),\dots,t_m(x_\gamma))$.

Proof. By (2.9) and Corollary 2.6, we get that $d^+_{\rho,\mu}(\Gamma,\varepsilon) - d^-_{\rho,\mu}(\Gamma,\varepsilon)$ equals

$$\begin{aligned} &\tfrac{1}{|F|} \sum_{\substack{\gamma \in \Lambda\backslash\Gamma \\ B \in F_1}} \chi_\rho(\gamma) \sum_{u \in (\Lambda^*_{\varepsilon,\mu})^B} e^{-2\pi i u\cdot b} \big(\chi_{L_{n-1}^{\sigma(u,x_\gamma)}} - \chi_{L_{n-1}^{-\sigma(u,x_\gamma)}}\big)(x_\gamma) \\ = \;&\tfrac{1}{|F|} \sum_{\substack{\gamma \in \Lambda\backslash\Gamma \\ B \in F_1}} \chi_\rho(\gamma) \sum_{u \in (\Lambda^*_{\varepsilon,\mu})^B} e^{-2\pi i u\cdot b}\sigma(u,x_\gamma) \big(\chi_{L^+_{n-1}} - \chi_{L^-_{n-1}}\big)(x_\gamma). \end{aligned}$$

Now, using (6.4) we get

$$d^+_{\rho,\mu}(\Gamma,\varepsilon) - d^-_{\rho,\mu}(\Gamma,\varepsilon) = \tfrac{(2i)^m}{|F|} \sum_{\substack{\gamma \in \Lambda\backslash\Gamma \\ B \in F_1}} \chi_\rho(\gamma)\, e_{\mu,\gamma,\sigma}(\delta_\varepsilon) \prod_{j=1}^m \sin t_j(x_\gamma).$$

By substituting this last expression in (2.17) the proposition follows. □

Remark 2.10. (i) In Proposition 3.4 we will give a very explicit expression for the eta series and will compute the eta invariant for a general flat manifold with holonomy group $\mathbb{Z}_2^k$. We shall see that we may have $\eta = 0$ or $\eta \neq 0$, depending on the spin structure.

(ii) Let $g \in \mathrm{Spin}(n)$. If n even, $\chi_{L_n^+}(g) - \chi_{L_n^-}(g) = \mathrm{Str}\, L_n(g)$, is the *supertrace* of $L_n(g)$. Furthermore, by Proposition 3.23 in [**BGV**], one has the expression:

$$\mathrm{Str}\, L_n(g) = i^{-n/2}\, \mathrm{sgn}(g)\, |\det(Id_{n-1} - \mu(g))|^{1/2}$$

where $\mathrm{sgn}(g) \in \{\pm 1\}$ is defined in [**BGV**] and μ is the covering map (1.3).

(iii) Note that some authors use $\frac{1}{2}(\eta(0) + d_0)$ instead of $\eta(0)$ as the definition of the η-invariant of (M,ε).

3. The twisted Dirac spectrum of $\mathbb{Z}_2^k$-manifolds.

In this section we shall look at the case of $\mathbb{Z}_2^k$-manifolds, a very rich class of flat manifolds. In this case, the holonomy group $F \simeq \mathbb{Z}_2^k$, but the holonomy action need not be diagonal in general. Already in the case when $k = 2$ it

is known that there are infinitely many indecomposable holonomy actions and infinitely many of these give rise to (at least) one Bieberbach group (see [**BGR**]). Giving a classification for $k \geq 3$ is a problem *of wild type*. The case $k = n-1$ corresponds to the so called generalized Hantzsche-Wendt manifolds, studied in [**MR**], [**RS**] and [**MPR**]. This class is still very rich and will be used in Section 4 to construct large families of Dirac isospectral manifolds pairwise non homeomorphic to each other.

If Γ has holonomy group $\mathbb{Z}_2^k$, then $\Gamma = \langle \gamma_1, \dots, \gamma_k, \Lambda \rangle$ where $\gamma_i = B_i L_{b_i}$, $B_i \in \mathrm{O}(n)$, $b_i \in \mathbb{R}^n$, $B_i \Lambda = \Lambda$, $B_i^2 = Id$ and $B_i B_j = B_j B_i$, for each $1 \leq i, j \leq k$. We shall see that, somewhat surprisingly, for these manifolds the multiplicity formulas in Theorem 2.5 take extremely simple forms (see (3.4) and (3.5)). This will allow to exhibit, in the next section, large Dirac isospectral sets. Also, we will be able to characterize all manifolds having asymmetric Dirac spectrum and to obtain very explicit expressions for the η-series and the η-invariant.

Let F_1 be as in (1.2). In the case of $\mathbb{Z}_2^k$-manifolds, F_1 is the set of $B \in \mathrm{O}(n) \cap r(\Gamma)$ such that B is conjugate in $\mathrm{O}(n)$ to the diagonal matrix $diag(-1, \dots, -1, 1)$. The next lemma will be very useful in the proof of Theorem 3.2, the main result in this section.

Lemma 3.1. *If Γ is a Bieberbach group with translation lattice Λ and holonomy group $\mathbb{Z}_2^k$ then the elements in F_1 can be simultaneously diagonalized in Λ, that is, there is a basis $f_1, \dots, f_n$ of Λ such that $Bf_j = \pm f_j$, $1 \leq j \leq n$, for any $BL_b \in \Gamma$ with $n_B = 1$. Furthermore Γ can be conjugated by some L_μ, $\mu \in \mathbb{R}^n$, to a group Γ' such that $2b \in \Lambda$ for any $BL_b \in \Gamma'$ with $n_B = 1$.*

Proof. Let $BL_b \in \Gamma$ with $n_B = 1$. Then $B\Lambda = \Lambda$ and it is a well known fact that Λ decomposes, with respect to the action of B, as $\Lambda = \Lambda_1 \oplus \Lambda_2$ where Λ_1 (resp. Λ_2) is a direct sum of integral subrepresentations of rank 1 (resp. 2). Here B acts diagonally on Λ_1, whereas Λ_2 is a direct sum of subgroups on which B acts by $J = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Now, it is not hard to check that since Γ is Bieberbach, then the orthogonal projection $2b^+ \in \Lambda$ of $2b$ onto $(\mathbb{R}^n)^B$ can not lie in Λ_2, otherwise some element of the form $BL_{b+\lambda}$, with $\lambda \in \Lambda$, would be of finite order (see [**DM**], Proposition 2.1). Thus, the multiplicity of the eigenvalue 1 for B is at least 1 on Λ_1. If furthermore $\Lambda_2 \neq 0$, then it would be at least 1 on Λ_2, hence $n_B \geq 2$, which is not possible since $B \in F_1$. Thus $\Lambda_2 = 0$ for every $B \in F_1$, therefore each such B can be diagonalized in a basis of Λ (see Lemma 3.3 in [**RS**] for a different proof).

Our next task is to show that this can be done simultaneously in Λ for all elements in F_1. To show this, we enumerate the elements in F_1: $B_1, \dots, B_r$. Let $f_1, \dots, f_n$ be a basis of Λ diagonalizing B_r. After reordering we may assume that $B_r f_n = f_n$ and $B_r f_j = -f_j$ for $1 \leq j \leq n-1$. Clearly f_n is orthogonal to f_j for $j < n$. Now, for $i = 1, \dots, r-1$, $B_i f_n = \pm f_n$ since B_i commutes with B_r, and actually $B_i f_n = -f_n$, since otherwise we would have $B_i = B_r$, since $B_i \in F_1$. Now, $B_1, \dots, B_{r-1} \in F_1$ and leave $\Lambda' := \mathbb{Z}\text{-span}\{f_1, \dots, f_{n-1}\}$ invariant. Hence by induction they can

be simultaneously diagonalized in some $\mathbb{Z}$-basis of Λ'. Putting this basis together with f_n we get a basis of Λ that diagonalizes $B_1, \dots, B_r$. The proof of the second assertion now follows in the same way as that of Lemma 1.4 in [**MR4**]. $\square$

We are now in a position to prove the main result in this section. We shall see that for a spin $\mathbb{Z}_2^k$-manifold, only the identity element, Id, and possibly one element $BL_b \in \Gamma$ with $n_B = 1$ can give a nonzero contribution to the multiplicity formulas.

We note that given $\gamma = BL_b \in \Gamma \smallsetminus \Lambda$, since $B^2 = Id$, then B is conjugate in $\mathrm{SO}(n)$ to a diagonal matrix of the form $\mathrm{diag}(\underbrace{-1, \dots, -1}_{2h}, 1, \dots, 1)$ with $1 \le h \le m$ where $m = [\frac{n}{2}]$, hence $\varepsilon(\gamma)$ is conjugate in $\mathrm{Spin}(n)$ to $\pm g_h$ for some $1 \le h \le m$, where

$$g_h := e_1 \cdots e_{2h} \in \mathrm{Spin}(n). \tag{3.1}$$

Moreover, if $n = 2m$ then $h < m$, since $B = -Id$ cannot occur for $BL_b \in \Gamma$, Γ being torsion-free. Thus, it follows that $g_h \in \mathrm{Spin}(n-1)$ and since g_h and $-g_h$ are conjugate (see Corollary 6.4), then we may take x_γ in Definition 2.3 to be equal to g_h, with h depending on γ. From now on in this section *we shall thus assume* that $x_\gamma = g_h$.

We will need the fact that (see Lemma 6.1), if $n = 2m$, then

$$\chi_{L_n^\pm}(g_h) = \begin{cases} \pm 2^{m-1} i^m & h = m \\ 0 & 1 \le h < m \end{cases} \tag{3.2}$$

and furthermore, for n even or odd one has

$$\chi_{L_n}(g_h) = 0 \qquad \text{for any } 1 \le h \le m. \tag{3.3}$$

Theorem 3.2. *Let (M_Γ, ε) be a spin $\mathbb{Z}_2^k$-manifold of dimension n, and let F_1 be as in (1.2).*

(i) If $F_1 = \emptyset$, then $Spec_{D_\rho}(M_\Gamma, \varepsilon)$ is symmetric and the multiplicities of the eigenvalues $\pm 2\pi\mu$, $\mu > 0$, of D_ρ are given by:

$$d^{\pm}_{\rho,\mu}(\Gamma, \varepsilon) = 2^{m-k-1}\, d_\rho\, |\Lambda^*_{\varepsilon,\mu}|. \tag{3.4}$$

(ii) If $F_1 \neq \emptyset$ then $Spec_{D_\rho}(M_\Gamma, \varepsilon)$ is asymmetric if and only if the following conditions hold: $n = 4r + 3$ and there exists $\gamma = BL_b$, with $n_B = 1$ and $\chi_\rho(\gamma) \neq 0$, such that $B_{|\Lambda} = -\delta_\varepsilon Id$. In this case, the asymmetric spectrum is the set of eigenvalues

$$\mathcal{A} = Spec^{\mathcal{A}}_{D_\rho}(M_\Gamma, \varepsilon) = \{\pm 2\pi\mu_j : \mu_j = (j + \tfrac{1}{2})\|f\|^{-1},\ j \in \mathbb{N}_0\}$$

where $\Lambda^B = \mathbb{Z} f$ and if we put $\sigma_\gamma := \sigma((f \cdot 2b) f, g_m)$ we have:

$$d^{\pm}_{\rho,\mu}(\Gamma, \varepsilon) = \begin{cases} 2^{m-k-1}\left(d_\rho\, |\Lambda^*_{\varepsilon,\mu}| \pm 2\sigma_\gamma (-1)^{r+j} \chi_\rho(\gamma)\right) & \mu = \mu_j, \\ 2^{m-k-1}\, d_\rho\, |\Lambda^*_{\varepsilon,\mu}| & \mu \neq \mu_j \end{cases} \tag{3.5}$$

If $Spec_{D_\rho}(M_\Gamma, \varepsilon)$ is symmetric then $d^{\pm}_{\rho,\mu}(\Gamma, \varepsilon)$ is given by (3.4).

(iii) The number of independent harmonic spinors is given by

$$d_{\rho,0}(\Gamma,\varepsilon) = \begin{cases} 2^{m-k} d_\rho & \text{if } \varepsilon_{|\Lambda} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

If $k > m$ then M_Γ has no spin structures of trivial type, hence M_Γ has no harmonic spinors. Furthermore, if M_Γ has exactly $2^m d_\rho$ harmonic spinors then $M_\Gamma = T_\Lambda$ and $\varepsilon = 1$.

Proof. We first note that the contribution of $Id \in F$ to the multiplicity formulas (2.8) and (2.9) is given by $2^{m-k-1} d_\rho\, e_{\mu,Id}(\delta_\varepsilon) = 2^{m-k-1} d_\rho\, |\Lambda^*_{\varepsilon,\mu}|$. Hence, when no element in F other than Id gives a nonzero contribution then (3.4) holds.

If $F_1 = \emptyset$, then, for any $\gamma = BL_b \in \Gamma \smallsetminus \Lambda$, we have $\varepsilon(\gamma) \sim g_h$ with $h < m$. Thus, $\chi_{L_n}(\varepsilon(\gamma)) = 0$ if n odd and $\chi^{\pm}_{L_n}(\varepsilon(\gamma)) = 0$ if n even, for any $\gamma \in \Gamma \smallsetminus \Lambda$. Thus, in this case, only $Id \in F$ contributes to (2.8) and (2.9) and hence (i) follows. This implies that, other than Id, only the elements in F_1 can give a nonzero contribution to the multiplicities and furthermore, this can happen only if n is odd.

Now, assume that $n = 2m+1$ and $F_1 \neq \emptyset$. If $\gamma = BL_b \in \Gamma \smallsetminus \Lambda$, with $n_B = 1$ (hence $x_\gamma = g_m$), we know by Lemma 3.1 that there exists a basis $f_1, \dots, f_n$ of Λ such that B is diagonal in this basis. After reordering the basis elements we may assume that

$$Bf_j = -f_j \text{ for } 1 \le j < n, \quad Bf_n = f_n, \quad b \equiv \frac{1}{2} f_n \mod \Lambda. \tag{3.6}$$

Let $f'_1, \dots, f'_n \in \Lambda^*$ be the dual basis of $f_1, \dots, f_n$. It is clear that also $Bf'_j = -f'_j$ for $1 \le j < n$ and $f'_n = \frac{f_n}{\|f_n\|^2}$, thus $Bf'_n = f'_n$.

Let as usual $\Lambda^*_\varepsilon = \Lambda^* + u_\varepsilon$, with $u_\varepsilon = \sum_j c_j f'_j$ and $c_j \in \{0, \frac{1}{2}\}$ for each j. If the contribution of B to (2.9) is non-trivial, then $(\Lambda^*_\varepsilon)^B \neq \emptyset$. Thus, there exists $u = \lambda' + u_\varepsilon$ with $\lambda' = \sum_j d_j f'_j$, $d_j \in \mathbb{Z}$ and such that $Bu = u$. This says that for $1 \le j \le n-1$, we have $c_j + d_j = -c_j - d_j$ and hence $c_j = 0$ for $1 \le j \le n-1$. On the other hand, c_n equals 0 or $\frac{1}{2}$, that is, $u_\varepsilon = 0$ or $u_\varepsilon = \frac{1}{2} f'_n$.

If $u_\varepsilon = 0$ we have $\Lambda^*_\varepsilon = \Lambda^*$ and thus $e^{2\pi i u\cdot b} = e^{-2\pi i u \cdot b}$ for any $u \in \Lambda^*$, since $b \in \frac{1}{2}\Lambda$, by Lemma 3.1. We claim that $\sum_u e^{-2\pi i u\cdot b}\, \chi_{L^{\pm\sigma(u,x_\gamma)}_{n-1}}(x_\gamma) = 0$ where the sum is taken over $u \in (\Lambda^*_{\varepsilon,\mu})^B$. Indeed, putting together the contributions of u and $-u$ in the expression above, by Remark 2.4 and (3.3) we get

$$e^{-2\pi i u\cdot b} \left(\chi_{L^{\pm\sigma(u,g_m)}_{n-1}} + \chi_{L^{\pm\sigma(-u,g_m)}_{n-1}} \right)(g_m) = e^{-2\pi i u \cdot b}\, \sigma(u, g_m)\, \chi_{L_{n-1}}(g_m) = 0.$$

Hence, we conclude that if $u_\varepsilon = 0$, the contribution of $\gamma = BL_b \in \Gamma \smallsetminus \Lambda$ to (2.9) is zero.

Now consider the case $u_\varepsilon = \frac{1}{2} f_n'$, that is, $(\Lambda_\varepsilon^*)^B = (\mathbb{Z}+\frac{1}{2})f_n'$. Hence, since $\delta_\varepsilon(\lambda) = e^{2\pi i u_\varepsilon\cdot\lambda}$, then

$$\varepsilon(L_{f_j}) = 1 \quad (1 \le j \le n-1), \qquad \text{and} \qquad \varepsilon(L_{f_n}) = -1. \tag{3.7}$$

Furthermore we note that, since

$$-1 = \varepsilon(L_{f_n}) = \varepsilon(\gamma^2) = \varepsilon(\gamma)^2 = (\pm e_1 \dots e_{2m})^2 = (-1)^m$$

it follows that m is necessarily odd, hence $m = 2r+1$ and $n = 4r+3$.

Now (3.6) and (3.7) say that only if $B_{|\Lambda} = -\delta_\varepsilon Id$ can $\gamma = BL_b$ give a nonzero contribution.

Furthermore, since $(\Lambda_\varepsilon^*)^B = (\mathbb{Z}+\frac{1}{2})f_n'$, then, for fixed $\mu > 0$, $(\Lambda_{\varepsilon,\mu}^*)^B \neq \emptyset$ if and only if $\mu = \mu_j := (j+\frac{1}{2})\|f_n\|^{-1}$ with $j \in \mathbb{N}_0$. For $\mu = \mu_j$, we have that $(\Lambda_{\varepsilon,\mu_j}^*)^B = \{\pm u_j\}$ where $u_j = (j+\frac{1}{2})f_n'$ with $j = \|f_n\|\mu_j - \frac{1}{2} \in \mathbb{N}_0$. Again, putting together the contributions of u_j and $-u_j$, we get that the sum over $(\Lambda_{\varepsilon,\mu}^*)^B$ in (2.9) equals

$$\begin{aligned} & e^{-2\pi i u_j\cdot b}\, \chi_{L_{n-1}^{\pm\sigma(u_j,g_m)}}(g_m) + e^{2\pi i u_j\cdot b}\, \chi_{L_{n-1}^{\pm\sigma(-u_j,g_m)}}(g_m) \\ = \;& \left(e^{-2\pi i u_j\cdot b} - e^{2\pi i u_j\cdot b}\right) \sigma(u_j,g_m)\, \chi_{L_{n-1}^{\pm}}(g_m) \\ = \;& \mp 2^m i^{m+1}\, \sigma(u_j,g_m) \sin(2\pi u_j\cdot b). \end{aligned}$$

where we have used that $\chi_{L_{n-1}^-}(g_h) = -\chi_{L_{n-1}^+}(g_h)$ by (3.2). If $\Lambda^B = \mathbb{Z}f$, then one has that $f = \pm f_n$. Now one verifies that $\sigma(u_j,g_m) = \sigma((f\cdot 2b)f, g_m) = \sigma_\gamma$. Hence,

$$\sigma(u_j,g_m)\sin(2\pi u_j\cdot b) = \sigma_\gamma \sin(\pi(j+\tfrac{1}{2})) = \sigma_\gamma\,(-1)^j$$

since $b \equiv \frac{1}{2} f_n \mod \Lambda$.

Since $m = 2r+1$, we finally get that the contribution of γ to the multiplicity of the eigenvalue $\pm 2\pi\mu_j$ is given by

$$\pm 2^{m-k}\sigma_\gamma\,(-1)^{r+j}\chi_\rho(\gamma)$$

The above shows that if an element $\gamma' = B'L_{b'} \in F_1$ gives a nonzero contribution, then $B'_{|\Lambda} = -\delta_\varepsilon Id$, hence $B' = B$, and $b' \equiv \frac{1}{2}f_n$, mod Λ. Since only Id and $\gamma = BL_b$ give a contribution to the multiplicity formula, this completes the proof of (ii).

Finally, the first assertion in (iii) follows immediately from (2.10) and the remaining assertions are direct consequences of the first. □

Remark 3.3. (i) Except for the very special case described in (ii) of the theorem, the twisted Dirac spectrum of $\mathbb{Z}_2^k$-manifolds is symmetric and the multiplicities are given by the simple formula (3.4). In this case, the multiplicities of Dirac eigenvalues for M_Γ with a spin structure ε are determined by the multiplicities for the covering torus T_Λ with the restricted spin structure $\varepsilon_{|\Lambda}$. Indeed, we have

$$d_{\rho,\mu}^{\pm}(\Gamma,\varepsilon) = 2^{-k}\, d_{\rho,\mu}^{\pm}(\Lambda,\varepsilon_{|\Lambda}).$$

(ii) Note that, for each fixed ρ, all spin $\mathbb{Z}_2^k$-manifolds (M_Γ, ε) having asymmetric Dirac spectrum and having the same covering torus $(T_\Lambda, \varepsilon_{|\Lambda})$ are D_ρ-isospectral to each other.

As an application of Theorem 3.2 we shall now compute the η-series and the η-invariant for any $\mathbb{Z}_2^k$-manifold.

Proposition 3.4. *Let (M_Γ, ε) be a spin $\mathbb{Z}_2^k$-manifold of odd dimension $n = 4r+3$ (thus $m = 2r+1$). If $Spec_{D_\rho}(M_\Gamma, \varepsilon)$ is asymmetric then, in the notation of Theorem 3.2, we have:*

$$\eta_{(\Gamma,\rho,\varepsilon)}(s) = (-1)^r\,\sigma_\gamma\,\chi_\rho(\gamma)\,2^{m-k+1}\,\frac{\|f\|^s}{(4\pi)^s}\big(\zeta(s,\tfrac14) - \zeta(s,\tfrac34)\big) \tag{3.8}$$

where $\zeta(s,\alpha) = \sum_{j=0}^{\infty}\frac{1}{(j+\alpha)^s}$ denotes the Riemann-Hurwitz zeta function for $\alpha \in (0,1]$ and $\sigma_\gamma \in \{\pm 1\}$ is as defined in (ii) of Theorem 3.2.

Therefore, $\eta_{(\Gamma,\rho,\varepsilon)}(s)$ has an analytic continuation to $\mathbb{C}$ that is everywhere holomorphic. Furthermore,

$$\eta_{(\Gamma,\rho,\varepsilon)}(0) = (-1)^r\sigma_\gamma\,\chi_\rho(\gamma)\,2^{m-k}, \tag{3.9}$$

$$\eta'_{(\Gamma,\rho,\varepsilon)}(0) = \big(4\log\Gamma(\tfrac14) + \log\|f\| - 3\log(2\pi)\big)\,\eta(0). \tag{3.10}$$

Proof. We shall use Theorem 3.2 in the case of the special spin structure when the spectrum is not symmetric, otherwise, $\eta(s) = 0$. We have

$$\eta_{(\Gamma,\rho,\varepsilon)}(s) = \frac{1}{(2\pi)^s}\sum_{j=0}^{\infty}\frac{d^+_{\rho,\mu_j}(\Gamma,\varepsilon) - d^-_{\rho,\mu_j}(\Gamma,\varepsilon)}{|\mu_j|^s}.$$

Now from formula (3.5) we have that

$$d^+_{\rho,\mu_j}(\Gamma,\varepsilon) - d^-_{\rho,\mu_j}(\Gamma,\varepsilon) = (-1)^{r+j}\,\sigma_\gamma\,\chi_\rho(\gamma)\,2^{m-k+1}.$$

Thus, if $Re(s) > n$

$$\begin{aligned}
\eta_{(\Gamma,\rho,\varepsilon)}(s) &= (-1)^r\sigma_\gamma\,\chi_\rho(\gamma)\,2^{m-k+1}\,\frac{||f||^s}{(2\pi)^s}\sum_{j=0}^{\infty}\frac{(-1)^j}{(j+\frac12)^s}\\
&= (-1)^r\sigma_\gamma\,\chi_\rho(\gamma)\,2^{m-k+1}\,\frac{||\frac12 f||^s}{(2\pi)^s}\left(\sum_{j=0}^{\infty}\frac{1}{(j+\frac14)^s} - \sum_{j=0}^{\infty}\frac{1}{(j+\frac34)^s}\right)\\
&= (-1)^r\sigma_\gamma\,\chi_\rho(\gamma)\,2^{m-k+1}\,\frac{||\frac12 f||^s}{(2\pi)^s}\big(\zeta(s,\tfrac14) - \zeta(s,\tfrac34)\big)
\end{aligned}$$

where $\zeta(s,\alpha)$ is the Riemann-Hurwitz function, for $\alpha \in (0,1]$. Now $\zeta(s,\alpha)$ extends to an everywhere holomorphic function except for a simple pole at $s = 1$, with residue 1 (see [**WW**], 13.13) hence formula (3.8) implies that $\eta(s)$ is everywhere holomorphic.

Furthermore, since $\zeta(0,\alpha) = \frac12 - \alpha$, by taking limit as $s \to 0$ in the above expression we get (3.9). Also, differentiating (3.8) and using that $\zeta'(0,a) = \log\Gamma(a) - \frac12\log(2\pi)$ (see [**WW**]) we obtain (3.10). $\square$

Remark 3.5. Note that if $(\rho, V) = (1, \mathbb{C})$, then for $\mathbb{Z}_2^k$-manifolds with $k \leq m$, one has that $\eta(0) \in 2\mathbb{Z}$. In particular the η-invariant of any $\mathbb{Z}_2$-manifold is an even integer. Indeed, in the asymmetric case, $\eta = \pm 2^{m-1} \in \mathbb{Z}$. In dimension $n = 3$, for $F \simeq \mathbb{Z}_2$, the proposition gives $\eta = \sigma_\gamma$. Take $\Gamma = \langle \gamma, L_\Lambda \rangle$ where $\gamma = \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} L_{\frac{e_3}{2}}$ and Λ the canonical lattice in $\mathbb{R}^3$. Then M_Γ has asymmetric spectrum only for the two spin structures $\varepsilon_+ = (1, 1, -1; e_1 e_2)$ and $\varepsilon_- = (1, 1, -1; -e_1 e_2)$. Then $\eta_{(\Gamma, \varepsilon_+)} = 1$ while $\eta_{(\Gamma, \varepsilon_-)} = -1$ as in [**Pf**].

4. Dirac isospectral manifolds

In this section we give examples of twisted Dirac isospectral flat manifolds that are pairwise non-homeomorphic to each other. In Examples 4.3, 4.4 and 4.5 we compare (twisted) Dirac isospectrality with other types of isospectrality, such as Laplace isospectrality on functions and on p-forms and length isospectrality with and without multiplicities (see Introduction). Two manifolds are $[L]$-isospectral (L-isospectral) if they have the same $[L]$-spectrum (L-spectrum). Obviously, $[L]$-isospectrality implies L-isospectrality.

As a consequence we will obtain the following results:

Theorem 4.1. *(i) There are families $\mathcal{F}$ of pairwise non-homeomorphic Riemannian manifolds mutually twisted Dirac isospectral that are neither Laplace isospectral on functions nor L-isospectral. Furthermore, $\mathcal{F}$ can be chosen so that:*

(a) Every $M \in \mathcal{F}$ has (no) harmonic spinors. (Ex. 4.3 (i)).

(b) All M's in $\mathcal{F}$ have the same p-Betti numbers for $1 \leq p \leq n$ and they are p-isospectral to each other for any p odd. (Ex. 4.3 (ii)).

(ii) There are pairs of non-homeomorphic spin manifolds that are $\Delta_{s,\rho}$-isospectral but not D_ρ-isospectral. (Ex. 4.4 (ii)).

(iii) There are pairs of spin manifolds that are Δ_p-isospectral for $0 \leq p \leq n$ and also $[L]$-isospectral which are D_ρ-isospectral, or not, depending on the spin structure. (Ex. 4.5 (i)).

(iv) There are pairs of spin manifolds that are D_ρ-isospectral and Δ_p-isospectral for $0 \leq p \leq n$ which are L-isospectral but not $[L]$-isospectral. (Ex. 4.5 (ii)).

Theorem 4.2. *There exists a family, with cardinality depending exponentially on n (or n^2), of pairwise non-homeomorphic Kähler Riemannian n-manifolds that are twisted Dirac isospectral to each other for many different spin structures. (Ex. 4.6, Rem. 4.7).*

To construct the examples, it will suffice to work with flat n-manifolds of diagonal type, having holonomy group $\mathbb{Z}_2^k$, $k = 1, 2$ and $n - 1$.

A Bieberbach group Γ is said to be of *diagonal type* if there exists an orthonormal $\mathbb{Z}$-basis $\{e_1, \ldots, e_n\}$ of the lattice Λ such that for any element $BL_b \in \Gamma$, $Be_i = \pm e_i$ for $1 \leq i \leq n$ (see [**MR4**]). Similarly, M_Γ is said to be of diagonal type, if Γ is so. If Γ is of diagonal type, after conjugation of Γ

by an isometry, it may be assumed that Λ is the canonical lattice and also that for any $\gamma = BL_b \in \Gamma$, one has that $b \in \frac{1}{2}\Lambda$ ([**MR4**], Lemma 1.4).

Let (M_Γ, ε) be a spin $\mathbb{Z}_2^k$-manifold where $\Gamma = \langle \gamma_1, \dots, \gamma_k, L_\Lambda \rangle$ and let $\lambda_1, \dots \lambda_n$ be a $\mathbb{Z}$-basis of Λ. If $\gamma = BL_b \in \Gamma$ we will fix a distinguished (though arbitrary) element in $\mu^{-1}(B)$, denoted by $u(B)$. Thus, $\varepsilon(\gamma) = \sigma\, u(B)$, where $\sigma \in \{\pm 1\}$ depends on γ and on the choice of $u(B)$.

The homomorphism $\varepsilon : \Gamma \to \mathrm{Spin}(n)$ is determined by its action on the generators $\lambda_1, \dots, \lambda_n, \gamma_1, \dots, \gamma_k$ of Γ. Hence, if we set $\delta_i := \varepsilon(L_{\lambda_i})$, we may identify the spin structure ε with the $n+k$-tuple

$$(\delta_1, \dots, \delta_n, \sigma_1 u(B_1), \dots, \sigma_k u(B_k)) \tag{4.1}$$

where σ_i is defined by the equation $\varepsilon(\gamma_i) = \sigma_i u(B_i)$, for $1 \le i \le k$.

Example 4.3 ($\mathbb{Z}_2$-manifolds)**.** Here we will give some large twisted Dirac isospectral sets of $\mathbb{Z}_2$-manifolds. Put $J := \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. For each $0 \le j, h < n$, define

$$B_{j,h} := \mathrm{diag}(\underbrace{J, \dots, J}_{j}, \underbrace{-1, \dots, -1}_{h}, \underbrace{1, \dots, 1}_{l}) \tag{4.2}$$

where $n = 2j + h + l$, $j + h \neq 0$ and $l \ge 1$. Then $B_{j,h} \in \mathrm{O}(n)$, $B_{j,h}^2 = Id$ and $B_{j,h} \in \mathrm{SO}(n)$ if and only if $j + h$ is even. Let $\Lambda = \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_n$ be the canonical lattice of $\mathbb{R}^n$ and for j, h as before define the groups

$$\Gamma_{j,h} := \langle B_{j,h} L_{\frac{e_n}{2}}, \Lambda \rangle. \tag{4.3}$$

We have that $(B_{j,h}+Id)\frac{e_n}{2} = e_n \in \Lambda \smallsetminus (B_{j,h}+Id)\Lambda$. Hence, by Proposition 2.1 in [**DM**], the $\Gamma_{j,h}$ are Bieberbach groups. In this way, if we set $M_{j,h} = \Gamma_{j,h} \backslash \mathbb{R}^n$, we have a family

$$\mathcal{F} = \{M_{j,h} = \Gamma_{j,h} \backslash \mathbb{R}^n \,:\, 0 \le j \le [\tfrac{n-1}{2}], 0 \le h < n - 2j,\, j + h \neq 0\} \tag{4.4}$$

of compact flat manifolds with holonomy group $F \simeq \mathbb{Z}_2$. The family $\mathcal{F}$ gives a system of representatives for the diffeomorphism classes of $\mathbb{Z}_2$-manifolds of dimension n (see Proposition 4.2 in [**MP**]).

We will make use the following result on the existence of spin structures (see Proposition 4.2 in [**MP**], and also Theorem 2.1 and Lemma 3.1):

$M_{j,h}$ has 2^{n-j} spin structures parametrized by the tuples $(\delta_1, \dots, \delta_n, \sigma) \in \{\pm 1\}^{n+1}$ satisfying:

$$\delta_1 = \delta_2,\ \cdots,\ \delta_{2j-1} = \delta_{2j} \qquad \text{and} \qquad \delta_n = (-1)^{\frac{j+h}{2}}. \tag{4.5}$$

Isospectrality on p-forms. We first review from [**MR3**] and [**MR4**] some results on spectra of Laplace operators on vector bundles over flat manifolds. If τ is a finite dimensional representation of $K = \mathrm{O}(n)$ and $G = I(\mathbb{R}^n)$ we form the vector bundle E_τ over $G/K \simeq \mathbb{R}^n$ associated to τ and consider the corresponding vector bundle $\Gamma \backslash E_\tau$ over $\Gamma \backslash \mathbb{R}^n = M_\Gamma$. As usual we denote by χ_τ and d_τ respectively, the character and the dimension of τ. Let $-\Delta_\tau$ be the connection Laplacian on this bundle.

We recall from [**MR4**], Theorem 2.1, that the multiplicity of the eigenvalue $4\pi^2\mu$ of $-\Delta_\tau$ is given by
(4.6)
$$d_{\tau,\mu}(\Gamma) = \tfrac{1}{|F|} \sum_{\gamma=BL_b\in\Lambda\backslash\Gamma} \chi_\tau(B)\, e_{\mu,\gamma} \quad \text{where} \quad e_{\mu,\gamma} = \sum_{\substack{v\,\in\,(\Lambda^*)^B \\ \|v\|^2\,=\,\mu}} e^{-2\pi i v\cdot b}.$$

If $\tau = \tau_p$, the p-exterior representation of $\mathrm{O}(n)$, then $-\Delta_{\tau_p}$ corresponds to the Hodge Laplacian acting on p-forms. In this case we shall write Δ_p, $\mathrm{tr}_p(B)$ and $d_{p,\mu}(\Gamma)$ in place of Δ_{τ_p} , $\mathrm{tr}\,\tau_p(B)$ and $d_{\tau_p,\mu}(\Gamma)$, respectively.

Thus, the p-Laplacian Δ_p, $0 \le p \le n$, has eigenvalues $4\pi^2\mu$ with multiplicities $d_{p,\mu}$ given by formula (4.6). Furthermore, for flat manifolds of diagonal type the traces $\mathrm{tr}_p(B)$ are given by integral values of the *Krawtchouk polynomials* $K_p^n(x)$ (see [**MR3**], Remark 3.6, and also [**MR4**]). Indeed, we have:

$$\text{(4.7)} \quad \mathrm{tr}_p(B) = K_p^n(n-n_B), \qquad \text{where } K_p^n(x) := \sum_{t=0}^{p} (-1)^t \binom{x}{t}\binom{n-x}{p-t}.$$

Note that $B_{j,h}$ and $B_{0,j+h}$ are conjugate in $\mathrm{GL}(n,\mathbb{R})$ hence $\mathrm{tr}_p(B_{j,h}) = \mathrm{tr}_p(B_{0,j+h}) = K_p^n(j+h)$. Thus,

$$\text{(4.8)} \qquad d_{p,\mu}(\Gamma_{j,h}) = \tfrac{1}{2}\big(\tbinom{n}{p}|\Lambda_{\sqrt{\mu}}| + K_p^n(j+h)\, e_{\mu,\gamma}(\Gamma_{j,h})\big).$$

Hence, the existence of integral zeros of $K_p^n(x)$ will imply p-isospectrality of some of the $M_{j,h} \in \mathcal{F}$. For some facts on integral zeros of $K_p^n(x)$ see [**KL**], p. 76, or Lemma 3.9 in [**MR4**]. The simplest are the so called trivial zeros, namely:

If n is even then $K_{\frac{n}{2}}^n(j) = K_j^n(\frac{n}{2}) = 0$ for any j odd. Also, $K_k^n(j) = 0$ if and only if $K_j^n(k) = 0$.

Thus for $n = 2m$, all manifolds $\{M_{j,h} : j+h \text{ is odd }\}$ are m-isospectral and all manifolds $\{M_{j,h} : j+h = m\}$ are p-isospectral for any p odd with $1 \le p \le n$. However, generically –i.e., for arbitrary p and n– the $M_{j,h}$ will not be p-isospectral to each other because the integral roots of Krawtchouk polynomials, aside from the trivial zeros, are very sporadic.

We claim that the manifolds in $\mathcal{F}$ are pairwise not isospectral on functions. Take $\mu = 1$. Then $\Lambda_1 = \{\pm e_1, \ldots, \pm e_n\}$ and $\Lambda_1^{B_{j,h}} = \{\pm e_{2j+h+1}, \ldots, \pm e_n\}$ thus $|\Lambda_1| = 2n$ and $|\Lambda_1^{B_{j,h}}| = 2(n-(2j+h)) = 2l$. Now, one checks that $e_{1,\gamma}(\Gamma_{j,h}) = 2(l-1) + 2(-1) = 2(l-2)$ and hence, from (4.8), we get

$$\text{(4.9)} \qquad d_{p,1}(\Gamma_{j,h}) = \tbinom{n}{p} n + K_p^n(j+h)(l-2).$$

Now consider $\mu = \sqrt{2}$. Then $\Lambda_{\sqrt{2}} = \{\pm(e_i \pm e_j) : 1 \le i < j \le n\}$ and $\Lambda_{\sqrt{2}}^{B_{j,h}} = \{\pm(e_{2i-1} + e_{2i}) : 1 \le i \le j\} \cup \{\pm(e_i \pm e_j) : l+1 \le i < j \le n\}$. Hence $|\Lambda_{\sqrt{2}}| = 4\binom{n}{2}$ and $|\Lambda_{\sqrt{2}}^{B_{j,h}}| = 2j + 4\binom{l}{2}$. One checks that $e_{\sqrt{2},\gamma}(\Gamma_{j,h}) =$

$2j-4(l-1)+(4\binom{l}{2}-4(l-1)) = 2j+2(l-1)(l-4)$. Again by (4.8), we get

$$d_{p,\sqrt{2}}(\Gamma_{j,h}) = 2\binom{n}{p}\binom{n}{2} + K_p^n(j+h)\big(j+(l-1)(l-4)\big). \tag{4.10}$$

In particular for $p=0$, since $K_0^n(j)=1$ for any j, we have

$$d_{0,1}(\Gamma_{j,h}) = n+l-2 \tag{4.11}$$

$$d_{0,\sqrt{2}}(\Gamma_{j,h}) = n(n-1)+j+(l-1)(l-4). \tag{4.12}$$

These multiplicities are sufficient to show that all $\mathbb{Z}_2$-manifolds in $\mathcal{F}$ are pairwise not isospectral. Indeed, if $M_{j,h}, M_{j',h'}$ are isospectral then $l=l'$ by (4.11), thus $2j+h=2j'+h'$. By (4.12), then $j=j'$ and hence $h=h'$.

Dirac isospectrality. We will give families of spin $\mathbb{Z}_2$-manifolds Dirac isospectral to each other.

We need to restrict ourselves to orientable manifolds, so consider the family

$$\mathcal{F}^+ = \{M_{j,h} \in \mathcal{F} : j+h \text{ is even}\}.$$

It will also be convenient to split $\mathcal{F}^+ = \mathcal{F}_0^+ \dot\cup \mathcal{F}_1^+$ where

$$\mathcal{F}_i^+ = \{M_{j,h} \in \mathcal{F}^+ : j+h \equiv 2i \mod (4)\}, \qquad i=0,1.$$

(i) We now define spin structures for $M_{j,h}$ in $\mathcal{F}^+$. By (4.5), $\delta_n = 1$ for $j+h \equiv 0\,(4)$ and $\delta_n=-1$ for $j+h\equiv 2\,(4)$. Hence, we take the spin structures

$$\varepsilon_{i,j,h} = (1,\ldots,1,(-1)^i;\sigma u(B_{j,h})), \qquad i=0,1, \tag{4.13}$$

for manifolds in $\mathcal{F}_i^+$, $i=0,1$, respectively. For simplicity, we will write ε_i for $\varepsilon_{i,j,h}$.

We claim that all the spin $\mathbb{Z}_2$-manifolds in $\tilde{\mathcal{F}}_0^+ := \{(M_{j,h},\varepsilon_0) : M_{j,h} \in \mathcal{F}_0^+\}$ are twisted Dirac isospectral to each other. Indeed, since ε_0 is a spin structure of trivial type, we know from Theorem 3.2 that the spectrum is symmetric and the multiplicities of the eigenvalues $\pm 2\pi\mu$ of D_ρ are given by

$$d^{\pm}_{\rho,\mu}(\Gamma_{j,h},\varepsilon_0) = 2^{m-2} d_\rho\, |\Lambda^*_{\varepsilon_0,\mu}| = 2^{m-2} d_\rho\, |\Lambda_\mu|$$

since $\Lambda^*_{\varepsilon_0} = \Lambda$. Note that all manifolds in $\tilde{\mathcal{F}}_0^+$ have $2^{m-1}d_\rho$ non-trivial harmonic spinors.

If $n \not\equiv 3(4)$, the spin manifolds in $\tilde{\mathcal{F}}_1^+ := \{(M_{j,h},\varepsilon_1) : M_{j,h} \in \mathcal{F}_1^+\}$ are Dirac isospectral to each other. The same happens with those in $\tilde{\mathcal{F}}_1^+ \smallsetminus \{M_{0,2m}\}$, for $n = 2m+1 \equiv 3(4)$. Indeed, in both cases, we have that $d^{\pm}_{\rho,\mu}(\Gamma_{j,h},\varepsilon_1) = 2^{m-2} d_\rho\, |\Lambda_{\varepsilon_1,\mu}|$, by Theorem 3.2. These manifolds do not have non-trivial harmonic spinors.

(ii) Note that, for every t, all $M_{t,0}, M_{t-1,1},\ldots,M_{0,t}$ have the same first Betti number. We recall from [**MP**], Proposition 4.1, that for $1 \le p \le n$

$$\beta_p(M_{j,h}) = \sum_{i=0}^{[\frac{p}{2}]} \binom{j+h}{2i}\binom{j+l}{p-2i}. \tag{4.14}$$

Hence, if $\beta_1(M_{j,h}) = \beta_1(M'_{j,h})$, then $\beta_p(M_{j,h}) = \beta_p(M'_{j,h})$ for any $p \geq 1$.

Now, take

$$\mathcal{F}_t = \{(M_{j,h}, \varepsilon) : M_{j,h} \in \mathcal{F}^+ \text{ and } j + h = t\} \tag{4.15}$$

for some fixed t even and ε as in (4.13). In this way $\mathcal{F}_t$ is a family of $t+1$ spin $\mathbb{Z}_2$-manifolds which are Dirac isospectral to each other all having the same p-Betti numbers for all $1 \leq p \leq n$. Moreover, if we take $n = 2t$ then all $M_{j,h} \in \mathcal{F}_t$ are p-isospectral for any p odd, by the comments after (4.8).

Example 4.4. Here we give a simple pair of non-homeomorphic spin $\mathbb{Z}_2^2$-manifolds that are (twisted) spinor Laplacian isospectral but not (twisted) Dirac isospectral. Let Λ be the canonical lattice in $\mathbb{R}^7$ and take the Bieberbach groups

$$\Gamma = \langle B_1 L_{b_1}, B_2 L_{b_2}, \Lambda\rangle, \qquad \Gamma' = \langle B_1 L_{b'_1}, B_2 L_{b'_2}, \Lambda\rangle$$

where $B_1 = diag(-1,-1,-1,-1,-1,-1,1)$, $B_2 = diag(-1,-1,1,1,1,1,1)$, $B'_1 = diag(-1,-1,-1,-1,1,1,1)$, $B'_2 = diag(1,1,-1,-1,-1,-1,1)$, and $b_1 = \frac{e_7}{2}$, $b_2 = \frac{e_1+e_3+e_7}{2}$, $b'_1 = \frac{e_7}{2}$, $b'_2 = \frac{e_2}{2}$ are in $\frac{1}{2}\Lambda$. Let $M_\Gamma = \Gamma\backslash\mathbb{R}^7$, $M_{\Gamma'} = \Gamma'\backslash\mathbb{R}^7$ be the associated $\mathbb{Z}_2^2$-manifolds.

By Theorem 2.1 in [**MP**], one can check that M_Γ and $M_{\Gamma'}$ respectively admit spin structures $\varepsilon, \varepsilon'$ with characters $\delta_\varepsilon = (\delta_1, \delta_2, \delta_1, \delta_4, \delta_5, \delta_6, -1)$ and $\delta_{\varepsilon'} = (\delta'_1, 1, \delta'_3, \delta'_4, \delta'_5, \delta'_6, 1)$, $\delta_i, \delta'_i \in \{\pm 1\}$.

Now, $F_1(\Gamma) = \{B_1\}$. If we take $\delta_\varepsilon = (1,1,1,1,1,1,-1)$, and if $\chi_\rho(B_1) \neq 0$, then (M_Γ, ε) has asymmetric Dirac spectrum. Thus, if $\mu_j = j + \frac{1}{2}$, $j \in \mathbb{N}_0$, the multiplicity of $\pm 2\pi\mu$ is given by

$$d^{\pm}_{\rho,\mu}(\Gamma, \varepsilon) = \begin{cases} d_\rho\,|\Lambda_{\varepsilon,\mu_j}| \pm 2(-1)^j \sigma_{\gamma_1} \chi_\rho(\gamma_1) & \mu = \mu_j \\ d_\rho\,|\Lambda_{\varepsilon,\mu}| & \mu \neq \mu_j. \end{cases}$$

On the other hand $F_1(\Gamma') = \emptyset$. Thus, $M_{\Gamma'}$ has symmetric Dirac spectrum with $d^{\pm}_{\rho,\mu}(\Gamma', \varepsilon') = d_\rho\,|\Lambda_{\varepsilon',\mu}|$.

Now take $\delta_{\varepsilon'} = (-1,1,1,1,1,1,1)$. Then (M_Γ, ε) and $(M_{\Gamma'}, \varepsilon')$ are $\Delta_{s,\rho}$-isospectral. Indeed, $d_0(\Gamma,\varepsilon) = d_0(\Gamma',\varepsilon') = 0$ and since $|J^-_\varepsilon| = |J^-_{\varepsilon'}| = 1$, for any $\mu > 0$, by Remark 2.2, we have

$$d_{\rho,\mu}(\Gamma,\varepsilon) = 2d_\rho\,|\Lambda_{\varepsilon,\mu}| = 2d_\rho\,|\Lambda_{\varepsilon',\mu}| = d_{\rho,\mu}(\Gamma',\varepsilon').$$

However, (M_Γ, ε) and $(M_{\Gamma'}, \varepsilon')$ are not D_ρ-isospectral for $Spec^A_{D_\rho}(\Gamma,\varepsilon) \neq \emptyset$ while $Spec^A_{D_\rho}(\Gamma',\varepsilon') = \emptyset$. Note that $Spec^S_{D_\rho}(\Gamma,\varepsilon) = Spec^S_{D_\rho}(\Gamma',\varepsilon')$.

Example 4.5. Here, we shall give two pairs of Δ_p-isospectral for $0 \leq p \leq n$ and L-isospectral 4-dimensional $\mathbb{Z}_2^2$-manifolds M, M', one pair being $[L]$-isospectral and the other not. These pairs will be twisted Dirac isospectral, or not, depending on the choices of the spin structures.

Consider the manifolds M_i, M'_i, $1 \leq i \leq 2$, where $M_i = \Gamma_i\backslash\mathbb{R}^4$, $M'_i = \Gamma'_i\backslash\mathbb{R}^4$ and $\Gamma_i = \langle\gamma_1, \gamma_2, \Lambda\rangle$, $\Gamma'_i = \langle\gamma'_1, \gamma'_2, \Lambda\rangle$ are as given in Table 2, where $\gamma_i = B_i L_{b_i}$, $\gamma'_i = B_i L_{b'_i}$, $i = 1,2$, $B_3 = B_1 B_2$, $b_3 = B_2 b_1 + b_2$, $b'_3 = B'_2 b'_1 + b'_2$ and $\Lambda = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_4$ is the canonical lattice. Furthermore, we take

$B_i = B_i'$. In all cases the matrices B_i are diagonal and are written as column vectors. We indicate the translation vectors b_i, b_i' also as column vectors, leaving out the coordinates that are equal to zero. We will also use the pair $\tilde{M}_2, \tilde{M}_2'$ of $\mathbb{Z}_2^2$-manifolds of dimension 6 obtained from the pair M_2, M_2' by adjoining the characters $(-1, 1, -1)$ and $(1, -1, -1)$ to B_i, $1 \leq i \leq 3$, and keeping b_i, b_i' unchanged.

Note that M_1, M_1' and $\tilde{M}_2, \tilde{M}_2'$ are non-orientable while M_2, M_2' not.

TABLE 2

	B_1	L_{b_1}	$L_{b_1'}$	B_2	L_{b_2}	$L_{b_2'}$	B_3	L_{b_3}	$L_{b_3'}$
$\{M_1, M_1'\}$	-1			1		1/2	-1		1/2
	-1			-1	1/2	1/2	1	1/2	1/2
	1		1/2	-1			-1		1/2
	1	1/2		1	1/2		1		

	B_1	L_{b_1}	$L_{b_1'}$	B_2	L_{b_2}	$L_{b_2'}$	B_3	L_{b_3}	$L_{b_3'}$
$\{M_2, M_2'\}$ $\{\tilde{M}_2, \tilde{M}_2'\}$	1			1		1/2	1		1/2
	1		1/2	1	1/2	1/2	1	1/2	
	1	1/2		-1			-1	1/2	
	-1			1	1/2		-1	1/2	
	-1			1			-1		
	1			-1			-1		

(i) By Example 3.3 in [**MR4**], the manifolds M_1, M_1' are Sunada isospectral (hence p-isospectral for $0 \leq p \leq n$) and $[L]$-isospectral.

By Theorem 2.1 in [**MP**], one can check that M_1 admits 2^4 spin structures ε_1 of the form $\varepsilon_1 = (\delta_1, -1, \delta_3, -1; \sigma_1 e_1 e_2, \sigma_2 e_2 e_3)$, and M_1' carries 2^3 spin structures ε_1' of the form $\varepsilon_1' = (-1, -1, -1, \delta_4', \sigma_1' e_1 e_2, \sigma_2' e_2 e_3)$ where $\delta_1, \delta_3, \delta_4', \sigma_1, \sigma_2, \sigma_1', \sigma_2' \in \{\pm 1\}$.

Since $F_1(\Gamma_1) = F_1(\Gamma_1') = \emptyset$, by (3.4) we have that $d^{\pm}_{\rho,\mu}(\Gamma_1, \varepsilon_1) = d_\rho \, |\Lambda_{\varepsilon_1,\mu}|$ and $d^{\pm}_{\rho,\mu}(\Gamma_1', \varepsilon_1') = d_\rho \, |\Lambda_{\varepsilon_1',\mu}|$.

Now, if we take $\delta_{\varepsilon_1} = (1, -1, 1, -1)$, we see that (M_1, ε_1) is not twisted Dirac isospectral to (M_1', ε_1') for any ε_1' because $|J^-_{\varepsilon_1}| = 2$ while $|J^-_{\varepsilon_1'}| \geq 3$ (see Remark 2.2). However, if we take $\varepsilon_1, \varepsilon_1'$ such that $\delta_{\varepsilon_1} = \delta_{\varepsilon_1'} = (-1, -1, -1, -1)$, then (M_1, ε_1) and (M_1', ε_1') are D_ρ-isospectral to each other.

(ii) By Example 3.4 in [**MR4**], the manifolds M_2, M_2' are Sunada isospectral (hence p-isospectral for $0 \leq p \leq n$ and L-isospectral) but not $[L]$-isospectral. In order to have orientable manifolds we add to M_2, M_2' the characters $(-1, 1, -1)$ and $(1, -1, -1)$. The pair $\tilde{M}_2, \tilde{M}_2'$ obtained has the same spectral properties as M_2, M_2'. This can be seen by proceeding as in Example 3.4 in [**MR4**].

Again, by Theorem 2.1 in [**MP**], we can check that $\tilde{M}_2$ has 2^5 spin structures, ε_2, with characters $\delta_{\varepsilon_2} = (\delta_1, 1, -1, -1, \delta_5, \delta_6)$, $\delta_1, \delta_5, \delta_6 \in \{\pm 1\}$; and

$\tilde{M}_2'$ has 2^6 spin structures, ε_2', with characters $\delta_{\varepsilon_2'} = (1, -1, \delta_3', \delta_4', \delta_5', \delta_6')$ where $\delta_3', \delta_4', \delta_5', \delta_6' \in \{\pm 1\}$.

Now, $d^{\pm}_{\rho,\mu}(\tilde{\Gamma}_2, \varepsilon_2) = d_\rho\, |\Lambda_{\varepsilon_2,\mu}|$ and $d^{\pm}_{\rho,\mu}(\tilde{\Gamma}_2', \varepsilon_2') = d_\rho\, |\Lambda_{\varepsilon_2',\mu}|$. As in (i), if we take spin structures $\varepsilon_2, \varepsilon_2'$ with $|J^-_{\varepsilon_2}| = |J^-_{\varepsilon_2'}|$, we see that $(\tilde{M}_2, \varepsilon_2)$ and $(\tilde{M}_2', \varepsilon_2')$ are D_ρ-isospectral to each other, whereas if we take $\varepsilon_2, \varepsilon_2'$ such that $|J^-_{\varepsilon_2}| \neq |J^-_{\varepsilon_2'}|$, then $(\tilde{M}_2, \varepsilon_2)$ and $(\tilde{M}_2', \varepsilon_2')$ are not D_ρ-isospectral.

Example 4.6. We now construct a large family of pairwise non-homeomorphic twisted Dirac isospectral flat $2n$-manifolds, with holonomy group $\mathbb{Z}_2^{n-1}$. We will apply the doubling procedure in [**JR**] or in [**BDM**] to the family of Hantzsche-Wendt manifolds (see [**MR**]).

We first recall some facts from [**MR**]. Let n be odd. A *Hantzsche-Wendt group* (or *HW group*) is an n-dimensional orientable Bieberbach group Γ with holonomy group $F \simeq \mathbb{Z}_2^{n-1}$ such that the action of every $B \in F$ diagonalizes on the canonical $\mathbb{Z}$-basis $e_1, \ldots, e_n$ of Λ. The holonomy group F can thus be identified to the diagonal subgroup $\{B : Be_i = \pm e_i,\ 1 \leq i \leq n,\ \det B = 1\}$ and $M_\Gamma = \Gamma\backslash\mathbb{R}^n$ is called a *Hantzsche-Wendt* (or *HW*) *manifold*.

We denote by B_i the diagonal matrix fixing e_i and such that $B_i e_j = -e_j$ (if $j \neq i$), for each $1 \leq i \leq n$. Clearly, F is generated by $B_1, B_2, \ldots, B_{n-1}$ and furthermore $B_n = B_1 B_2 \ldots B_{n-1}$.

Any HW group has the form $\Gamma = \langle \gamma_1, \ldots, \gamma_{n-1}, L_\lambda : \lambda \in \Lambda\rangle$, where $\gamma_i = B_i L_{b_i}$ for some $b_i \in \mathbb{R}^n$, $1 \leq i \leq n-1$, and where it may be assumed that the components b_{ij} of b_i satisfy $b_{ij} \in \{0, \frac{1}{2}\}$, for $1 \leq i, j \leq n$. Also, since $(\Lambda^p(\mathbb{R}^n))^{\mathrm{F}} = 0$ for any $1 \leq p \leq n-1$, it follows that all HW manifolds are rational homology spheres. We further recall that, as shown in [**MR**] (by using a rather small subfamily $\mathcal{H}_1$), the cardinality h_n of the family of all HW groups under isomorphism satisfies $h_n > \frac{2^{n-3}}{n-1}$. Moreover, the cardinality of the set of Laplace isospectral, pairwise non-isomorphic, *pairs* of HW groups grows exponentially with n.

Relative to the spin condition, it is easy to verify using (ε_1) in [**MP**], (2.3), that the manifolds in this family are generally not spin. Indeed, we now show that no HW manifold with $n = 4k+1$ is spin. We note that $\gamma_k^2 = L_{e_k}$ for each $1 \leq k \leq n$. Hence a spin structure ε must satisfy $\delta_k = \varepsilon(L_{e_k}) = \varepsilon(\gamma_k^2) = \varepsilon(\gamma_k)^2 = (\pm e_1 \ldots e_{k-1} e_{k+1} \ldots e_n)^2 = 1$, since $n = 4k+1$. Thus, it follows that $\varepsilon_{|\Lambda} = Id$. On the other hand $(\gamma_i\gamma_j)^2 = L_\lambda$ for some $\lambda \neq 0$ and since $\varepsilon(\gamma_i\gamma_j) = \pm e_i e_j$ it follows that $\varepsilon(L_\lambda) = \varepsilon(\pm e_i e_j)^2 = -1$, a contradiction.

In the case $n = 4k+3$, $k > 0$, we shall show that no HW manifold in the family $\mathcal{H}_1$ (see [**MR**]) is spin. More generally, assume that there are three consecutive generators of Γ, $\gamma_i = B_i L_{b_i}$ with $b_i = L_{(e_i + e_{i-1})/2}$, $i \geq 2$. Thus, if $\gamma := \gamma_i\gamma_{i+1}\gamma_{i+2} = B_i B_{i+1} B_{i+2} L_{(e_{i-1}+e_{i+2})/2}$, then we have $\gamma^2 = \gamma_{i+2}^2 = L_{e_{i+2}}$. This gives a contradiction since $\varepsilon(\gamma)^2 = 1$ (the multiplicity of the eigenvalue -1 for $B_i B_{i+1} B_{i+2}$ is $4k$) while $\varepsilon(\gamma_{i+2})^2 = -1$ (the multiplicity

of the eigenvalue -1 for B_{i+2} is $4k+2$). Actually, J.P. Rossetti has recently shown to us a proof (that still uses the criterion in [**MP**]) that no HW manifold can admit a spin structure.

We will now consider for any HW group Γ, the group $\mathrm{d}\Gamma$, defined by the doubling construction in [**JR**] or [**BDM**], namely $\mathrm{d}\Gamma = \langle dBL_{(b,b)}, L_{(\lambda_1,\lambda_2)} : BL_b \in \Gamma, \lambda_1, \lambda_2 \in \Lambda\rangle$ where $dB := \left[\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix}\right]$. This yields a Bieberbach group of dimension $2n$, with the same holonomy group $\mathbb{Z}_2^{n-1}$ as Γ, and with the additional property that the associated manifold $M_{\mathrm{d}\Gamma}$ is Kähler. The reason why we use $\mathrm{d}\Gamma$ in place of Γ is that $M_{\mathrm{d}\Gamma}$ is always spin (see [**MP**], Remark 2.4).

We will need the following facts:

(i) If Γ *is a HW group, then* $M_{\mathrm{d}\Gamma}$ *admits* 2^{n-1} *spin structures of trivial type.*

(ii) If Γ *runs through all HW groups, all manifolds* $M_{\mathrm{d}\Gamma}$ *endowed with spin structures of trivial type are twisted Dirac isospectral to each other.*

(iii) If Γ, Γ' *are non-isomorphic HW groups then* $\mathrm{d}\Gamma, \mathrm{d}\Gamma'$ *are non-isomorphic Bieberbach groups.*

(iv) Two HW groups Γ, Γ' *are Laplace isospectral if and only if* $\mathrm{d}\Gamma$ *and* $\mathrm{d}\Gamma'$ *are Laplace isospectral.*

Now, (i) and (ii) are direct consequences of Remark 2.4 in [**MP**] and of (i) of Theorem 3.2, respectively.

Proof of (iii). This assertion follows by an argument very similar to that given in the proof of Proposition 1.5 in [**MR**]. We shall only sketch it. An isomorphism between $\mathrm{d}\Gamma$ and $\mathrm{d}\Gamma'$ must be given by conjugation by CL_c with $C \in \mathrm{GL}(2n, \mathbb{R})$ and $c \in \mathbb{R}^{2n}$. Now $C(\Lambda \oplus \Lambda) = \Lambda \oplus \Lambda$ implies $C \in \mathrm{GL}(2n, \mathbb{Z})$ and furthermore for each $1 \leq i \leq n-1$,

$$CL_c dB_i L_{(b_i,b_i)} L_{-c} C^{-1} = dB_{\sigma(i)} L_{(b'(\sigma(i), b'(\sigma(i))}$$

where $\sigma \in S_n$. Thus $CdB_iC^{-1}L_{C((b_i,b_i)+(dB_i - Id)c)}$. In particular, this implies that $CdB_iC^{-1} = dB_{\sigma(i)}$ for each $1 \leq i \leq n$, with $\sigma \in S_n$. Thus, there is an $n \times n$ permutation matrix P such that $D := CdPC^{-1}$ commutes with dB_i for each i, thus D preserves $\mathbb{Z}e_i \oplus \mathbb{Z}e_{n+i}$ for each i. It is easy to see that conjugation by such D yields an automorphism of $\mathrm{d}\Gamma'$. Thus, conjugation by $DCL_c = dPL_c$ takes $\mathrm{d}\Gamma$ onto $\mathrm{d}\Gamma'$ isomorphically and furthermore

$$\begin{aligned} dPL_c dB_i L_{(b_i,b_i)} L_{-c} dP^{-1} &= d(PB_iP)^{-1} L_{dP((b_i,b_i)+(dB_i - Id)c)} \\ &= dB_{\sigma(i)} L_{(b'_i,b'_i)}. \end{aligned}$$

This implies that $c = (c_1, c_1)$, mod Λ, with $c_1 \in \frac{1}{4}\Lambda$ and hence, conjugation by PL_{c_1} gives an isomorphism between Γ and Γ', a contradiction.

Proof of (iv). Since HW groups are of diagonal type, then Γ, Γ' are Laplace isospectral if and only if they are Sunada isospectral, that is, if they have the same Sunada numbers (see [**MR3**], [**MR4**]). We claim that this is the case if and only if $\mathrm{d}\Gamma$ and $\mathrm{d}\Gamma'$ are Sunada isospectral to each other.

Indeed, we recall that for $0 \le t \le s \le n$, and Γ of diagonal type, the *Sunada numbers* of Γ are defined by

$$c_{d,t}(\Gamma) := \left|\left\{BL_b \in \Gamma : n_B = d \text{ and } n_B(\tfrac{1}{2}) = t\right\}\right|.$$

where, for $BL_b \in \Gamma$, $n_B := \dim(\mathbb{R}^n)^B = |\{1 \le i \le n : Be_i = e_i\}|$ and $n_B(\frac{1}{2}) := |\{1 \le i \le n : Be_i = e_i \text{ and } b \cdot e_i \equiv \frac{1}{2} \mod (\mathbb{Z})\}|$. Now, it is clear from the definitions that $c_{2s,2t}(\mathrm{d}\Gamma) = c_{s,t}(\Gamma)$ for each $0 \le t \le s \le n$ and $c_{u,v}(\mathrm{d}\Gamma) = 0$, if either u or v is odd. This clearly implies that Γ and Γ' have the same Sunada numbers if and only if $\mathrm{d}\Gamma$ and $\mathrm{d}\Gamma'$ do so, that is, if and only if $\mathrm{d}\Gamma$ and $\mathrm{d}\Gamma$ are Sunada-isospectral to each other.

Thus, for each n odd, by (i), (ii), (iii), (iv), the above construction yields a family, of cardinality that depends exponentially on n, of Kähler flat manifolds of dimension $2n$, all pairwise non-homeomorphic and all twisted Dirac isospectral to each other, having only $2d_\rho$ harmonic spinors for every trivial spin structure chosen (see (iii) of Theorem 3.2). Within this family, by (iv) and [**MR**], there are exponentially many pairs that are Sunada isospectral, hence p-isospectral for all p. However, generically, two such manifolds will not be p-isospectral for any value of p (see for instance [**MR2**] in the case $n = 7$).

We note that if we repeat the doubling procedure then the set of all $M_{\mathrm{d}^2\Gamma}$, with Γ a HW group, is an exponential family of hyperkähler manifolds with the same spectral properties as the family $M_{\mathrm{d}\Gamma}$, but now having $2^{n+1}d_\rho$ harmonic spinors for every spin structure of trivial type chosen.

Remark 4.7. If one looks at the family of all flat manifolds with holonomy group $\mathbb{Z}_2^{n-1}$ (see [**RS**]) for $n = 4r + 3$, then it is shown in [**MPR**] that a subfamily of this family has cardinality $2^{\frac{(n-1)(n-2)}{2}}$. If we apply the doubling procedure to this family, one shows that considerations (i), . . . , (iv) in Example 4.6 remain valid, hence one obtains a family of twisted Dirac isospectral, pairwise non homeomorphic, $2n$-manifolds of cardinality $2^{\frac{(n-1)(n-2)}{2}}$.

5. Eta invariants of $\mathbb{Z}_p$-manifolds

In this section we shall illustrate the results in Section 2 by using the expression (2.18) of the eta series, to compute explicitly the eta invariant of certain flat p-manifolds with cyclic holonomy group $\mathbb{Z}_p$, $p = 4r + 3$, p prime, for the two existing spin structures. In [**SS**], the authors give an expression for the eta invariant and harmonic spinors of this family (without assuming p to be prime), in terms of the solutions of certain equations in congruences. They give explicit values for $p = 3, 7$. Here we shall give an explicit expression for the eta invariant of this family in terms of Legendre symbols and special values of trigonometric functions. At the end we give a table with the values of η for any prime $p \le 503$. For $n = 3$ they coincide with those computed in [**Pf**] (and [**SS**]). Our formulas for the eta series

involve trigonometric sums and resemble those obtained in [**HZ**] to compute the G-index of elliptic operators for certain low dimensional manifolds.

We thus assume that F is cyclic of order $p = 4r+3$ (i.e. $m = 2r+1$) with p prime. Let $\Gamma = \langle BL_b, L_\Lambda\rangle$. Here $\Lambda = \sum_{i=1}^{p} \mathbb{Z}e_i$, B is of order p with $B(e_i) = e_{i+1}$ for $1 \le i \le p-2$, $B(e_{p-1}) = -\sum_{i=1}^{p-1} e_i$, $B(e_p) = e_p$ and $b = \frac{1}{p}e_p$.

Now, there exists $D \in \mathrm{GL}_p(\mathbb{R})$, $De_p = e_p$ such that $C := DBD^{-1} \in \mathrm{SO}(p)$. Thus, $\Gamma_p := D\Gamma D^{-1} = \langle \gamma = CL_{\frac{e_p}{p}}, D\Lambda\rangle$ is a Bieberbach group and $M_p = \Gamma_p\backslash\mathbb{R}^p$ is an orientable $\mathbb{Z}_p$-manifold of dimension p. The vectors $f_i := De_i$ for $1 \le i \le p-1$, and $f_p = e_p$ give a $\mathbb{Z}$-basis of $D\Lambda$.

Since the eigenvalues of B and C are the p-roots of unity: $\{e^{\frac{2\pi ik}{p}} : 1 \le k \le p-1\}$, then we can assume by further conjugation in $\mathrm{SO}(p)$ that $C = x_0(\frac{2\pi}{p}, \dots, \frac{2m\pi}{p})$ (see (1.4)) with $m = \frac{p-1}{2}$.

We now note that Theorem 2.1 in [**MP**], stated for $\mathbb{Z}_2^k$-manifolds, also holds for $\mathbb{Z}_n$-manifolds, replacing condition (ε_1): $\varepsilon(\gamma^2) = \varepsilon(\gamma)^2$, by (ε_1'): $\varepsilon(\gamma^n) = u_B^n$ for any $u_B \in \mathrm{Spin}(n)$ such that $\mu(u_B) = B$ and keeping condition (ε_2) in [**MP**], i.e. $\varepsilon(L_{(B-Id)(\lambda)}) = 1$ for any $\lambda \in \Lambda$. Thus, using conditions (ε_1') and (ε_2) one can see that $M_p := \Gamma_p\backslash\mathbb{R}^p$ has exactly two spin structures, $\varepsilon_1, \varepsilon_2$, given, on the generators of Γ, by

$$\varepsilon_1(L_{f_j}) = \varepsilon_2(L_{f_j}) = 1 \ (1 \le j \le p-1), \quad \varepsilon_1(L_{e_p}) = 1,\ \varepsilon_2(L_{e_p}) = -1,$$
$$\varepsilon_1(\gamma) = (-1)^{r+1}x(\tfrac{\pi}{p}, \tfrac{2\pi}{p}, \dots, \tfrac{(2r+1)\pi}{p}), \quad \varepsilon_2(\gamma) = (-1)^{r}x(\tfrac{\pi}{p}, \tfrac{2\pi}{p}, \dots, \tfrac{(2r+1)\pi}{p}),$$

in the notation of (1.4). We can now state:

Theorem 5.1. *Let $p = 4r+3$ be a prime and let Γ_p, $\varepsilon_1, \varepsilon_2$ be as above. Then, the eta series of $M_p = \Gamma_p\backslash\mathbb{R}^p$, for ε_1 and ε_2 are respectively given by:*

$$\begin{aligned} \eta_{(\Gamma_p,\rho,\varepsilon_1)}(s) &= \tfrac{-2\,\chi_\rho(\gamma)}{\sqrt{p}\,(2\pi p)^s} \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \sum_{l=1}^{p-1} \sin(\tfrac{2l\pi k}{p})\,\zeta(s, \tfrac{l}{p}), \\ \eta_{(\Gamma_p,\rho,\varepsilon_2)}(s) &= \tfrac{-2\,\chi_\rho(\gamma)}{\sqrt{p}\,(2\pi p)^s} \sum_{k=1}^{p-1} (-1)^k \left(\frac{k}{p}\right) \sum_{l=0}^{p-1} \sin(\tfrac{(2l+1)\pi k}{p})\,\zeta(s, \tfrac{2l+1}{2p}), \end{aligned} \tag{5.1}$$

and the eta invariants have the expressions

$$\begin{aligned} \eta_{\rho,\varepsilon_1} &= \tfrac{-2\,\chi_\rho(\gamma)}{\sqrt{p}} \sum_{k=1}^{\frac{p-1}{2}} \left(\frac{k}{p}\right) cot(\tfrac{k\pi}{p}), \\ \eta_{\rho,\varepsilon_2} &= \tfrac{-2\,\chi_\rho(\gamma)}{\sqrt{p}} \sum_{k=1}^{\frac{p-1}{2}} (-1)^k \left(\frac{k}{p}\right) cosec(\tfrac{k\pi}{p}), \end{aligned} \tag{5.2}$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

Note. The theorem shows that one can have a spin structure of trivial type (ε_1) with asymmetric Dirac spectrum, in contrast with the situation in the case of holonomy group $\mathbb{Z}_2^k$.

Expressions (5.1) and (5.2) can be simplified further by using identities in number theory. In particular one shows that the eta invariants take integer values. We plan to get deeper into this question in a sequel to this paper.

Proof. We will compute the different ingredients in formula (2.18) for the eta function for the two given spin structures. For $1 \leq k \leq p-1$ we have $(\Lambda_\varepsilon^*)^{B^k} = \mathbb{Z}e_p$ for ε_1 while $(\Lambda_\varepsilon^*)^{B^k} = (\mathbb{Z}+\frac{1}{2})e_p$ for ε_2. In both cases $(\Lambda_{\varepsilon,\mu}^*)^{B^k} = \{\pm\mu e_p\}$. Thus, $\mu = j$ for ε_1 and $\mu = j - \frac{1}{2}$ for ε_2 with $j \in \mathbb{N}$. We take $x_{\gamma^k} = \varepsilon_i(\gamma)^k$, for $i = 1, 2$. This implies that $\sigma(e_p, x_{\gamma^k}) = 1$ for $\varepsilon_1, \varepsilon_2$.

Since $\gamma^k = B^k L_{\frac{k}{p}e_p}$, by (2.7) and Remark 2.4, for $1 \leq k \leq p-1$, we have

$$e_{\mu,\gamma^k,\sigma}(\delta_\varepsilon) = e^{-2\pi i \frac{\mu k}{p}}\sigma(\mu e_p, x_{\gamma^k}) + e^{2\pi i\frac{\mu k}{p}}\sigma(-\mu e_p, x_{\gamma^k}) = -2i\sin(\tfrac{2\pi\mu k}{p}).$$

Thus, up to the factor $-2i$, the sums over $\mu \in \frac{1}{2\pi}\mathcal{A}$ corresponding to $\gamma = \gamma^k$ in (2.18) for ε_1 and ε_2 are respectively equal to

$$\sum_{j=1}^{\infty}\frac{\sin(\frac{2j\pi k}{p})}{j^s} = \sum_{l=1}^{p-1}\sin(\tfrac{2l\pi k}{p})\sum_{j=1}^{\infty}\frac{1}{(pj+l)^s} = \frac{1}{p^s}\sum_{l=1}^{p-1}\sin(\tfrac{2l\pi k}{p})\zeta(s,\tfrac{l}{p}),$$

(5.3)

$$\sum_{j=0}^{\infty}\frac{\sin(\frac{(2j+1)\pi k}{p})}{(j+\frac{1}{2})^s} = \frac{1}{p^s}\sum_{l=0}^{p-1}\sin(\tfrac{(2l+1)\pi k}{p})\zeta(s,\tfrac{2l+1}{2p}),$$

where $\zeta(s,\alpha) = \sum_{j=0}^{\infty}\frac{1}{(j+\alpha)^s}$ denotes the Riemann-Hurwitz zeta function for $\alpha \in (0,1]$.

We now compute the product of sines in (2.18) in both cases. We note that $t_j(x_{\gamma^k})$ (see (1.4)) depends on γ^k and also on $\varepsilon_1, \varepsilon_2$. We have, for ε_h, $h = 1, 2$:

$$\begin{aligned}\Pi_{j=1}^{\frac{p-1}{2}}\sin t_j(x_{\gamma^k}) &= (-1)^{k(r+h)}\,\Pi_{j=1}^{\frac{p-1}{2}}\sin(\tfrac{\pi jk}{p})\\ &= (-1)^{k(r+h)}\,\Pi_{j=1}^{\frac{p-1}{2}}(-1)^{[\frac{jk}{p}]}\,\Pi_{j=1}^{\frac{p-1}{2}}\sin(\tfrac{\pi j}{p})\\ &= (-1)^{k(r+h)}(-1)^{s_p(k)}\frac{\sqrt{p}}{2^{2r+1}},\end{aligned}$$

where we have put $s_p(k) = \sum_{j=1}^{\frac{p-1}{2}}[\frac{jk}{p}]$ and used the identities

$$\sin(\pi z) = \frac{\pi}{\Gamma(z)\Gamma(1-z)}, \quad (2\pi)^{\frac{p-1}{2}}\Gamma(z) = p^{z-\frac{1}{2}}\Gamma(\tfrac{z}{p})\Gamma(\tfrac{z+1}{p})\cdots\Gamma(\tfrac{z+p-1}{p}).$$

Now, if $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol, then (see [**Ap**], Theorems 9.6 and 9.7) we have, since $p = 4r+3$,

$$(-1)^{s_p(k)} = (-1)^{(k-1)(\frac{p^2-1}{8})}\left(\frac{k}{p}\right) = (-1)^{(k-1)(r+1)}\left(\frac{k}{p}\right).$$

In this way, we obtain:

$$\Pi_{j=1}^{\frac{p-1}{2}} \sin t_j(x_{\gamma^k}) = \begin{cases} (-1)^{(r+1)}\, 2^{-2r-1}\sqrt{p}\left(\frac{k}{p}\right) & \text{for } \varepsilon_1 \\ (-1)^{(r+1)}\, 2^{-2r-1}(-1)^k\sqrt{p}\left(\frac{k}{p}\right) & \text{for } \varepsilon_2. \end{cases} \tag{5.4}$$

Now, starting from (2.18) and using (5.3) and (5.4) we finally arrive at the expressions for the η-series of (M_p, ε_h), $h = 1, 2$, given in (5.1).

We now compute the eta invariants. Using that $\zeta(0,\alpha) = \frac{1}{2} - \alpha$ ([**WW**], 13.21), together with the fact that $\sum_{l=1}^{p-1}\sin(\frac{2l\pi k}{p}) = \sum_{l=0}^{p-1}\sin(\frac{(2l+1)\pi k}{p}) = 0$, for every $1 \le k \le p-1$, we see that the sums over l in the expressions (5.1), when evaluated at $s = 0$ are respectively equal to

$$-\frac{1}{p}\sum_{l=1}^{p-1} l\sin(\tfrac{2l\pi k}{p}), \text{ for } \varepsilon_1, \quad -\frac{1}{2p}\sum_{l=0}^{p-1}(2l+1)\sin(\tfrac{(2l+1)\pi k}{p}), \text{ for } \varepsilon_2. \tag{5.5}$$

We claim that

$$\sum_{l=1}^{p-1} l\sin(\tfrac{2l\pi k}{p}) = -\tfrac{p}{2}\cot(\tfrac{k\pi}{p}), \quad \sum_{l=0}^{p-1} l\sin\left(\tfrac{(2l+1)\pi k}{p}\right) = -\tfrac{p}{2}\operatorname{cosec}(\tfrac{k\pi}{p}). \tag{5.6}$$

Indeed, by differentiating the identity $\frac{1}{2} + \sum_{l=1}^{p-1}\cos(lx) = \frac{\sin((p-\frac{1}{2})x)}{2\sin(\frac{x}{2})}$, we get

$$-\sum_{l=1}^{p-1} l\sin(lx) = \frac{(2p-1)\cos((p-\frac{1}{2})x)}{4\sin(\frac{x}{2})} - \frac{\cos(\frac{x}{2})\sin((p-\frac{1}{2})x)}{4\sin^2(\frac{x}{2})}.$$

Evaluating both sides at $x = \frac{2k\pi}{p}$ yields the first equality in (5.6).

To verify the second identity we first note that

$$\sum_{l=1}^{p-1} l\cos(\tfrac{2l\pi k}{p}) = p\sum_{l=1}^{\frac{p-1}{2}}\cos(\tfrac{2l\pi k}{p}) = -\tfrac{p}{2}.$$

Using this expression together with the first identity in (5.6) we have

$$\begin{aligned}\sum_{l=0}^{p-1} l\sin\left(\tfrac{(2l+1)\pi k}{p}\right) &= \cos(\tfrac{\pi k}{p})\cot(\tfrac{\pi k}{p})(-\tfrac{p}{2}) + \sin(\tfrac{\pi k}{p})(-\tfrac{p}{2}) \\ &= -\tfrac{p}{2}\operatorname{cosec}(\tfrac{\pi k}{p}).\end{aligned}$$

Hence, from (5.1), (5.5) and (5.6) we obtain

$$\eta_{\varepsilon_1} = \frac{-1}{\sqrt{p}}\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)\cot(\tfrac{k\pi}{p}), \quad \eta_{\varepsilon_2} = \frac{-1}{\sqrt{p}}\sum_{k=1}^{p-1}(-1)^k\left(\frac{k}{p}\right)\operatorname{cosec}(\tfrac{k\pi}{p}).$$

We finally note that the contributions of k and $p-k$ to the above sums are equal to each other, that is:

$$\Big(\frac{k}{p}\Big)\cot(\tfrac{k\pi}{p}) = \Big(\frac{p-k}{p}\Big)\cot(\tfrac{(p-k)\pi}{p})$$
$$(-1)^k\Big(\frac{k}{p}\Big)\mathrm{cosec}(\tfrac{k\pi}{p}) = (-1)^{p-k}\Big(\frac{p-k}{p}\Big)\mathrm{cosec}(\tfrac{(p-k)\pi}{p}).$$

These identities can be easily verified using that

$$\Big(\frac{p-k}{p}\Big) = \Big(\frac{-k}{p}\Big) = \Big(\frac{-1}{p}\Big)\Big(\frac{k}{p}\Big) = (-1)^{\frac{p-1}{2}}\Big(\frac{k}{p}\Big) = -\Big(\frac{k}{p}\Big)$$

where in the last equality we have used that $p \equiv 3(4)$.

Taking into account this observation, we obtain the expressions (5.2) in the proposition. □

We now look at the simplest case when $p = 3$, first considered in [**Pf**]. We have $r = 0$ and $\varepsilon_1 = (1, 1, 1, -x(\frac{\pi}{3})) = (1, 1, 1, x(\frac{\pi}{3}+\pi))$, $\varepsilon_2 = (1, 1, -1, x(\frac{\pi}{3}))$ (in the notation of Section 4, see (4.1)). Since $\big(\frac{1}{p}\big) = 1$ we obtain

$$\begin{aligned}\eta_{\varepsilon_1}(0) &= \tfrac{-2}{\sqrt{3}}\big(\big(\tfrac{1}{3}\big)\cot(\tfrac{\pi}{3})\big) = -\tfrac{2}{\sqrt{3}}\cdot\tfrac{1}{\sqrt{3}} = -\tfrac{2}{3},\\ \eta_{\varepsilon_2}(0) &= \tfrac{-2}{\sqrt{3}}\big(-\big(\tfrac{1}{3}\big)\mathrm{cosec}(\tfrac{\pi}{3})\big) = \tfrac{2}{\sqrt{3}}\cdot\tfrac{2}{\sqrt{3}} = \tfrac{4}{3}.\end{aligned}$$

It is reassuring to see that the values are in coincidence with those in [**Pf**], after all these calculations.

To conclude this section, we shall give explicitly the eta invariants for all p-manifolds in the family, $p = 4r + 3$ prime, $7 \le p \le 503$, obtained with the help of a computer, using formulas (5.2), in the case $(\rho, V) = (1, \mathbb{C})$. We also give some values of $d_0(\varepsilon_1)$, the dimension of the space of harmonic spinors using (2.10) and (6.5). Note that $d_0(\varepsilon_2) = 0$ by Theorem 2.5. We summarize the information in the following table:

r	$p = 4r + 3$	η_{ε_1}	η_{ε_2}	$d_0(\varepsilon_1)$
0	3	$-\frac{2}{3}$	$\frac{4}{3}$	0
1	7	-2	0	2
2	11	-2	4	2
4	19	-2	4	26
5	23	-6	0	90
7	31	-6	0	1058
10	43	-2	4	48770
11	47	-10	0	178482
14	59	-6	12	9099506
16	67	-2	4	128207978
17	71	-14	0	483939978

r	$p = 4r + 3$	η_{ε_1}	η_{ε_2}
19	79	-10	0
20	83	-6	12
25	103	-10	0
26	107	-6	12
31	127	-10	0
32	131	-10	20
34	139	-6	12
37	151	-14	0
40	163	-2	4
41	167	-22	0

r	p	η_{ε_1}	η_{ε_2}
44	179	-10	20
47	191	-26	0
49	199	-18	0
52	211	-6	12
55	223	-14	0
56	227	-10	20
59	239	-6	12
62	251	-30	28
65	263	-26	0
67	271	-22	0

r	p	η_{ε_1}	η_{ε_2}
70	283	-6	12
76	307	-6	12
77	311	-38	0
82	331	-6	12
86	347	-10	20
89	359	-38	0
91	367	-18	0
94	379	-6	12
95	383	-34	0
104	419	-18	36

r	p	η_{ε_1}	η_{ε_2}
107	431	-42	0
109	439	-30	0
110	443	-10	20
115	463	-14	0
116	467	-14	28
119	479	-50	0
121	487	-14	0
122	491	-18	36
124	499	-6	12
125	503	-42	0

6. APPENDIX: SOME FACTS ON SPIN GROUPS AND SPIN REPRESENTATIONS

In this appendix we collect some facts on conjugacy classes on $\mathrm{Spin}(n)$ and on spin representations that are used in the body of the paper. For standard facts on spin geometry we refer to [**LM**] or [**Fr2**].

We consider (L, S), an irreducible complex representation of the Clifford algebra $\mathbb{C}l(n)$, restricted to $\mathrm{Spin}(n)$. The complex vector space S has dimension 2^m with $m = [\frac{n}{2}]$ and is called the spinor space. We have that $\mathrm{S} = \sum_{I \subset \{1,\dots,m\}} \mathrm{S}_{\lambda_I}$ where S_{λ_I} denotes the weight space corresponding to the weight

$$\lambda_I = \frac{1}{2}\Big(\sum_{i=1}^{m} \varepsilon_i\Big) - \sum_{i \in I} \varepsilon_i. \tag{6.1}$$

Here ε_j is given on the Lie algebra of T, $\mathfrak{t}$, by $\varepsilon_j(\sum_{k=1}^m c_k e_{2k-1} e_{2k}) = 2ic_j$. All weights have multiplicity 1. If n is odd, then (L, S) is irreducible for $\mathrm{Spin}(n)$ and is called the *spin representation.* If n is even, then the subspaces

$$\mathrm{S}^+ := \sum_{|I|=even} \mathrm{S}_{\lambda_I}, \qquad \mathrm{S}^- := \sum_{|I|=odd} \mathrm{S}_{\lambda_I}. \tag{6.2}$$

are $\mathrm{Spin}(n)$-invariant and irreducible of dimension 2^{m-1}. If $L^\pm$ denote the restricted action of L on $\mathrm{S}^\pm$ then $(L^\pm, \mathrm{S}^\pm)$ are called the *half-spin representations* of $\mathrm{Spin}(n)$. We shall write (L_n, S_n) (resp. $(L_n^\pm, \mathrm{S}_n^\pm)$) for (L, S), (resp. $(L^\pm, \mathrm{S}^\pm)$), when we wish to specify the dimension. We have the following wellknown facts:

$$\begin{array}{ll} (L^\pm_{n\,|\mathrm{Spin}(n-1)}, \mathrm{S}^\pm_n) \simeq (L_{n-1}, \mathrm{S}_{n-1}) & \text{if } n \text{ is even,} \\ (L_{n|\mathrm{Spin}(n-1)}, \mathrm{S}_n) \simeq (L^+_{n-1}, \mathrm{S}^+_{n-1}) \oplus (L^-_{n-1}, \mathrm{S}^-_{n-1}) & \text{if } n \text{ is odd.} \end{array} \tag{6.3}$$

The next lemma gives the values of the characters of L_n and $L_n^\pm$ on elements of T.

Lemma 6.1. *If $n = 2m$, then*

$$\chi_{L_n^{\pm}}(x(t_1, \dots, t_m)) = 2^{m-1}\Big(\prod_{j=1}^{m} \cos t_j \pm i^m \prod_{j=1}^{m} \sin t_j\Big). \tag{6.4}$$

If $n = 2m$ or $n = 2m + 1$, then

$$\chi_{L_n}(x(t_1, \dots, t_m)) = 2^m \prod_{j=1}^{m} \cos t_j. \tag{6.5}$$

Proof. Assume first that $n = 2m$ is even and proceed by induction on m. For $m = 1$, (6.4) clearly holds. Assume it holds for $n = 2m$. Set $I_m = \{1, \dots, m\}$. Now, $\chi_{L^+_{n+2}}(x(t_1, \dots, t_{m+1}))$ equals

$$\begin{aligned}
&= e^{i(\sum_1^{m+1} t_j)} \sum_{\substack{I \subset I_{m+1} \\ |I| \text{ even}}} e^{-2i\sum_{j\in I} t_j} \\
&= e^{it_{m+1}} e^{i(\sum_1^m t_j)} \Big(\sum_{\substack{I \subset I_m \\ |I| \text{ even}}} e^{-2i\sum_{j\in I} t_j} \;+\; e^{-2it_{m+1}} \sum_{\substack{I \subset I_m \\ |I| \text{ odd}}} e^{-2i\sum_{j\in I} t_j}\Big) \\
&= e^{it_{m+1}} \chi_{L_n^+}\big(x(t_1, \dots, t_m)\big) \;+\; e^{-it_{m+1}} \chi_{L_n^-}\big(x(t_1, \dots, t_m)\big) \\
&= 2^{m-1}\Big(e^{it_{m+1}}\big(\prod_{j=1}^{m} \cos t_j + i^m \prod_{j=1}^{m} \sin t_j\big) + e^{-it_{m+1}}\big(\prod_{j=1}^{m} \cos t_j - i^m \prod_{j=1}^{m} \sin t_j\big)\Big) \\
&= 2^m \Big(\prod_{j=1}^{m+1} \cos t_j + i^{m+1} \prod_{j=1}^{m+1} \sin t_j\Big).
\end{aligned}$$

The calculation for $\chi_{L^-_{n+2}}$ is analogous. By adding $\chi_{L_n^+}(x(t_1, \dots, t_m))$ and $\chi_{L_n^-}(x(t_1, \dots, t_m))$ we get the asserted expression for $\chi_{L_n}(x(t_1, \dots, t_m))$ if $n = 2m$. If $n = 2m + 1$, then $\chi_{L_n}(x(t_1, \dots, t_m)) = \chi_{L_{n-1}}(x(t_1, \dots, t_m))$, hence the result follows. □

The next lemma gives some useful facts on conjugacy classes of elements in Spin(n). We include a proof for completeness.

Lemma 6.2. *Let $x, y \in Spin(n-1)$ be conjugate in $Spin(n)$.*
(i) If n is even, then x, y are conjugate in $Spin(n-1)$.
(ii) If n is odd, then y is conjugate to x or to $-e_1 x e_1$ in $Spin(n-1)$.

Proof. If $n = 2m$ is even, the restriction map from the representation ring $R(\mathrm{Spin}(2m))$ to $R(\mathrm{Spin}(2m-1))$ is onto, hence the assertion in the lemma follows.

If $n = 2m+1$, we may assume that $x = x(t_1, \dots, t_m), y = x(t'_1, \dots, t'_m)$ lie in the maximal torus T, where $x(t_1, \dots, t_m) = \prod_{j=1}^m (\cos t_j + \sin t_j e_{2j-1} e_{2j})$.

Now, if x and y are conjugate in $\mathrm{Spin}(2m+1)$, then $\mu(x), \mu(y)$ are conjugate in $\mathrm{SO}(2m+1)$ and this implies that, after reordering, we must have $t_i' = \pm t_i$, for $1 \le i \le m$.

Furthermore if $1 \le j \le m$ we have

$$\begin{aligned} e_{2j-1}e_n x(t_1,\dots,t_m)(e_{2j-1}e_n)^{-1} &= e_{2j-1}x(t_1,\dots,t_j,\dots,t_m)(e_{2j-1})^{-1} \\ &= x(t_1,\dots,-t_j,\dots,t_m). \end{aligned}$$

Hence, if $x = x(t_1,\dots,t_m)$, then

$$(e_{2j-1}e_{2k-1})\, x\, (e_{2j-1}e_{2k-1})^{-1} = x(t_1,\dots,-t_j,\dots,-t_k,\dots,t_m)$$

for $1 \le j,k \le m$. Thus, for fixed $t_1,\dots,t_m$, among the elements of the form $x(\pm t_1,\dots,\pm t_m)$, there are at most two conjugacy classes in $\mathrm{Spin}(2m)$ represented by $x(\pm t_1, t_2 \dots, t_m)$ and $x(-t_1, t_2 \dots, t_m) = -e_1 x(t_1, t_2 \dots, t_m) e_1$.

Now by Lemma 6.1, we have that

$$\chi_{L^{\pm}_{n-1}}(x(t_1,\dots,t_m)) = 2^{m-1}\Big(\prod_{j=1}^{m} \cos t_j \pm i^m \prod_{j=1}^{m} \sin t_j\Big).$$

This implies that $x(t_1, t_2 \dots, t_m)$ and $x(-t_1, t_2 \dots, t_m)$ are not conjugate unless $t_j \in \pi\mathbb{Z}$ for some j. On the other hand, if this is the case, then clearly $e_1 e_{2j-1} \in \mathrm{Spin}(n-1)$ conjugates one element into the other. This completes the proof of the lemma. □

Remark 6.3. The lemma shows that generically, if n is odd, $x(t_1, t_2 \dots, t_m)$ and $x(-t_1, t_2 \dots, t_m)$ are conjugate in $\mathrm{Spin}(n)$ but not in $\mathrm{Spin}(n-1)$.

We now consider the special case when $t_i \in \frac{\pi}{2}\mathbb{Z}$ for all i, then $\mu(x)$ has order 2 (or 1). Set $g_h = e_1 e_2 \dots e_{2h-1} e_{2h} \in \mathrm{Spin}(n)$ for $1 \le h \le m = [\frac{n}{2}]$. Thus $g_h = x(\underbrace{\tfrac{\pi}{2},\dots,\tfrac{\pi}{2}}_{h}, 0,\dots,0)$ and $-g_h = x(\underbrace{-\tfrac{\pi}{2},\tfrac{\pi}{2},\dots,\tfrac{\pi}{2}}_{h}, 0,\dots,0)$.

Corollary 6.4. *If $h < m$, then g_h and $-g_h$ are conjugate in Spin$(n-1)$. If $h = m$ and $n = 2m$, then $\chi_{L^{\pm}_n}(g_m) = \pm 2^{m-1} i^m$, hence g_m and $-g_m$ are not conjugate. If $h = m$ and $n = 2m+1$, then $\chi_{L^{+}_{n-1}}(\pm g_m) = \pm 2^{m-1} i^m$, hence g_m and $-g_m$ are conjugate in Spin(n) but not in Spin$(n-1)$.*

Proof. The first assertion in the corollary follows immediately from the proof of Lemma 6.2. The remaining assertions are clear in light of Lemma 6.1. Indeed, for $h < m$ we have $e_1 e_n g_h (e_1 e_n)^{-1} = -g_h$. □

Recall that for any $u \in \mathbb{R}^n \smallsetminus \{0\}$, left Clifford multiplication by u on S is given by $u \cdot w = L(u)(w)$ for $w \in$ S. We fix $\langle\,,\rangle$ an inner product on S such that $L(u)$ is skew Hermitian, hence $\langle\,,\rangle$ is *Spin*(n)-invariant. Note that $L(u)^2 = -\|u\|^2 Id$. Hence, S decomposes $\mathrm{S} = \mathrm{S}_u^+ \oplus \mathrm{S}_u^-$, where $\mathrm{S}_u^{\pm}$ denote the eigenspaces, of dimension 2^{m-1}, of $L(u)$ with eigenvalues $\mp i\|u\|$.

Definition 6.5. If $u \in \mathbb{R}^n \smallsetminus \{0\}$ set

$$\mathrm{Spin}(n-1,u) := \{g \in \mathrm{Spin}(n) : gug^{-1} = u\}. \tag{6.6}$$

Clearly $\mathrm{Spin}(n-1, e_n) = \mathrm{Spin}(n-1)$ and for general u, if $h_u \in \mathrm{Spin}(n)$ is such that $h_u u {h_u}^{-1} = \|u\| e_n$, then $h_u \mathrm{Spin}(n-1, u) h_u^{-1} = \mathrm{Spin}(n-1)$. We note that for any $g \in \mathrm{Spin}(n-1, u)$, $L(g)$ commutes with $L(u)$, hence $L(g)$ preserves the eigenspaces $\mathrm{S}_u^{\pm}$.

The following lemma is used in the proof of Theorem 2.5.

Lemma 6.6. *Let Spin$(n-1, u)$ be as in (6.6). Then as Spin$(n-1)$-modules: $S_{e_n}^{\pm} \simeq (L_{n-1}^{\pm}, S_{n-1}^{\pm})$ if n is odd and $S_{e_n}^{\pm} \simeq (L_{n-1}, S_{n-1})$ if n is even. As Spin$(n-1, u)$-modules we have that $S_u^{\pm} = L(h_u) S_{n-1}^{\pm}$, if n is odd, and $S_u^{\pm} = L(h_u) S_{n-1}$, if n is even, with action given by $L(h_u x h_u^{-1}) = L(h_u) L(x) L(h_u^{-1})$ for any $x \in$ Spin$(n-1)$.*

Proof. $L(e_n)$ commutes with the action of $\mathrm{Spin}(n-1)$ on S_n and, on the other hand, $\mathrm{S}_n = \mathrm{S}_{n-1}^{+} \oplus \mathrm{S}_{n-1}^{-}$ as a $\mathrm{Spin}(n-1)$-module.

If n is odd, then $\mathrm{S}_{n-1}^{\pm}$ are inequivalent representations of $\mathrm{Spin}(n-1)$, hence $L(e_n)\mathrm{S}_{n-1}^{\pm} = \mathrm{S}_{n-1}^{\pm}$ and by Schur's lemma, $L(e_n)$ must act by multiplication by a scalar on each of them. By using the explicit description of L in [**Kn**], p. 286–288, one verifies that $L(e_n)$ acts by $\mp i$ on $\mathrm{S}_{n-1}^{\pm}$, that is $\mathrm{S}_{e_n}^{\pm} \simeq \mathrm{S}_{n-1}^{\pm}$.

If n is even, then $\mathrm{S}_n^{\pm}$ both restrict to S_{n-1} as $\mathrm{Spin}(n-1)$-modules. Since the $\pm i$-eigenspaces of $L(e_n)$ are stable by $\mathrm{Spin}(n-1)$, they must both be equivalent to S_{n-1}.

The remaining assertions are easily verified. □

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